Math 223s, Fall 2010
HW2

Due date: Friday, November 19, 1pm at start of class

Definition $(D, \leq)$ is said to be a *directed set* if $\leq$ is a transitive[1], reflexive[2], and for any two $d_1, d_2 \in D$ there exists some d with

$$d \geq d_1,$$

and

$$d \geq d_2.$$

For $(D, \leq)$ a directed set, we say that $((M_d)_{d\in D}, (j_{d_1,d_2})_{d_1 \leq d_2})$ is a *directed system over* D if each

$$j_{d,d'} M_d \to M_{d'}$$

and for $d_1 \leq d_2 \leq d_3$ we have

$$j_{d_1,d_3} = j_{d_2,d_3} \circ j_{d_1,d_2}.$$

Q1 In the above situation, let X be the set of pairs (a, d), where $d \in D, a \in M_d$. Set

$$(a, d) \sim (a', d')$$

if there is some $d^* \geq d, d'$ with

$$j_{d,d^*(a)} = j_{d',d^*(a')}.$$

Show that $\sim$ forms an equivalence relation on X.

Definition We then let $\mathrm{DirLim}_D(M_d, j_{d,d'})$ be the set of $\sim$ equivalence classes in X. This is often called the *direct limit* of the models.

Q2 Show there if define at each d_0 the map

$$j_{d_0,\infty} : M_d \to \mathrm{DirLim}_D(M_d, j_{d,d'}),$$

$$a \mapsto [(a, d_0)]_\sim,$$

then each $j_{d_0,\infty}$ is well defined and for $d_1 \leq d_2$ we have

$$j_{d_1,\infty} = j_{d_2,\infty} \circ j_{d_1,d_2}.$$

Q3 Now assume additionally that each M_d is a model for the language of $\in$, and each $j_{d,d'}$ is injective and respects the relation $\in$ (i.e. $M_d \models a \in b$ if and only if $M_{d'} \models j_{d,d'}(a) \in j_{d,d'}(b)$). We then set

$$[(a, d)]_\sim \in^{\mathrm{DirLim}_D(M_d, j_{d,d'})} [(a', d')]_\sim$$

if there is some $d^* \geq d, d'$ with

$$j_{d,d^*}(a) \in^{M_{d^*}} j_{d'd^*}(b).$$

Show that in this indicated structure each $j_{d,\infty}$ is injective and respects the relation $\in$.

Q4 Show that if each $j_{d,d'}$ is Σ_n, then so is each $j_{d,\infty}$.

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[1] $\forall d_1, d_2, d_3((d_1 \leq d_2 \wedge d_2 \leq d_3) \Rightarrow d_1 \leq d_3)$
[2] $\forall d(d \leq d)$

Q5 Show that it is possible to have each M_d well founded but the direct limit ill founded.

Q6 Suppose now we have a rival directed system over D, denote it $((N_d)_{d\in D}, (i_{d_1,d_2})_{d_1\leq d_2})$. Suppose that at each d we have a map

$$\rho_d : M_d \to N_d$$

which have the commutativity property that at $d \leq d'$

$$i_{d,d'} \circ \rho_d = \rho_{d'} \circ j_{d,d'}.$$

Show there is a unique

$$\rho_\infty : \mathrm{DirLim}_D(M_d, j_{d,d'}) \to \mathrm{DirLim}_D(N_d, i_{d,d'})$$

with the property that each d

$$i_{d,\infty} \circ \rho_d = \rho_\infty \circ j_{d,\infty}.$$

Q7 Show that in the above situation, if each of the $j_{d,d'}, i_{d,d'}$ maps are Σ_n-elementary and each ρ_d is Σ_{n+1}-elementary, then ρ_∞ is Σ_{n+1}-elementary.