HW 1, 223s, Fall 2010

Due date: Friday, October 29, 1pm (start of class)

Here is just one long problem, with various parts, most of which require at least one of the earlier steps.

Q1 Let δ be a regular cardinal and $C \subset \delta$ a club of indiscernibles for L_δ.

(i) (1pt) Show that if $f : \delta \to \delta$ is definable (without the use of parameters) over L_δ and regressive on C ($\forall \alpha \in C(f(\alpha) < \alpha)$) then it is constant on C.

(ii) (2pts) Show that if $\alpha \in C$, then α is a cardinal in L_δ. (Hint: If not, then no $\alpha \in C$ will be a cardinal, and we can define over L_δ a regressive function from C, $\alpha \mapsto |\alpha|^L$, which will be constant by the last exercise. But then we can let $g(\alpha)$ be the $<_L$-least element of $(\mathcal{P}(|\alpha| \times |\alpha|))^L$ which gives a well ordering of $|\alpha|$ isomorphic to α. Show that g will constant on C, and derive a contradiction.)

(iii) (2pt) Show that every element of C will be inacessible in L. (Assume the last exercise.)

(iv) (2pts) Show that if $\alpha < \delta$, then L_α is an elementary substructure of L_δ.

Now additionally assume that L_δ is the Skolem Hull of C. Fix some $\kappa \in C$. Fix some order preserving function $\pi : C \to C$ with the property that $\pi(\beta) = \beta$ for $\beta < \kappa$ in C but $\pi(\kappa) > \kappa$. Then define

$$j : L_\delta \to L_\delta$$

by

$$j(f(\gamma_1, ..., \gamma_n)) \mapsto f(\pi(\gamma_1), ..., \pi(\gamma_n))$$

for $f : L_\delta \to L_\delta$ a function definable (without use of parameters) over L_δ and $\gamma_1, ..., \gamma_n \in C$.

(v) (3pts) Show that j is well defined and elementary.

Q2 (No formal credit, but if you want something more to sharpen your teeth on give it a shot)

In the situation of (v) above, show that if we define μ on $\mathcal{P}(\kappa)^L$ by

$$\mu = \{A \subset \kappa : A \in L, \kappa \in j(A)\},$$

then μ gives an L-ultrafilter on κ and $\text{Ult}_\mu(L_\delta, L)$ is well founded.

Conclude that for every ordinal α we have $\text{Ult}_\mu(L_\alpha, L)$ well founded. And then conclude that at every regular γ there is a club of indiscernibles for L_γ.

(These conclusions probably require some of the results proved in class, and for the second one you may need the *proof* of how we went from elementary embeddings of L to $0^\sharp$.)