HW 1, 223s, Fall 2010

Due date: Friday, October 29, 1pm (start of class)

Here is just one long problem, with various parts, most of which require at least one of the earlier steps.

Q1 Let \( \delta \) be a regular cardinal and \( C \subset \delta \) a club of indiscernibles for \( L_\delta \).

(i) (1pt) Show that if \( f : \delta \to \delta \) is definable (without the use of parameters) over \( L_\delta \) and regressive on \( C \) (\( \forall \alpha \in C(f(\alpha) < \alpha) \)) then it is constant on \( C \).

(ii) (2pts) Show that if \( \alpha \in C \), then \( \alpha \) is a cardinal in \( L_\delta \). (Hint: If not, then no \( \alpha \in C \) will be a cardinal, and we can define over \( L_\delta \) a regressive function from \( C \), \( \alpha \mapsto |\alpha|^L \), which will be constant by the last exercise. But then we can let \( g(\alpha) \) be the \( <_L \)-least element of \( (P(|\alpha| \times |\alpha|))^L \) which gives a well ordering of \( |\alpha| \) isomorphic to \( \alpha \). Show that \( g \) will constant on \( C \), and derive a contradiction.)

(iii) (2pt) Show that every element of \( C \) will be inaccessible in \( L \). (Assume the last exercise.)

(iv) (2pts) Show that if \( \alpha < \delta \), then \( L_\alpha \) is an elementary substructure of \( L_\delta \).

Now additionally assume that \( L_\delta \) is the Skolem Hull of \( C \). Fix some \( \kappa \in C \). Fix some order preserving function \( \pi : C \to C \) with the property that \( \pi(\beta) = \beta \) for \( \beta < \kappa \) in \( C \) but \( \pi(\kappa) > \kappa \). Then define

\[ j : L_\delta \to L_\delta \]

by

\[ j(f(\gamma_1, ..., \gamma_n)) \mapsto f(\pi(\gamma_1), ..., \pi(\gamma_n)) \]

for \( f : L_\delta \to L_\delta \) a function definable (without use of parameters) over \( L_\delta \) and \( \gamma_1, ..., \gamma_n \in C \).

(v) (3pts) Show that \( j \) is well defined and elementary.

Q2 (No formal credit, but if you want something more to sharpen your teeth on give it a shot)

In the situation of (v) above, show that if we define \( \mu \) on \( P(\kappa)^L \) by

\[ \mu = \{ A \subset \kappa : A \in L, \kappa \in j(A) \}, \]

then \( \mu \) gives an \( L \)-ultrafilter on \( \kappa \) and \( \text{Ult}_{\mu}(L_\delta, L) \) is well founded.

Conclude that for every ordinal \( \alpha \) we have \( \text{Ult}_{\mu}(L_\alpha, L) \) well founded. And then conclude that at every regular \( \gamma \) there is a club of indiscernibles for \( L_\gamma \).

(These conclusions probably require some of the results proved in class, and for the second one you may need the proof of how we went from elementary embeddings of \( L \) to \( 0^\sharp \).)