Math 223d, Fall 2009

On the completion of a group with respect to a left invariant metric

I will restrict my remarks to Polish groups, simply for streamlining purposes. It will probably be clear that many of these are much more general.

Here are some facts and definitions. The first of these will most likely be proved in class.

Fact 0.1 *Let G be a Polish group. Let d be a left invariant compatible metric. Let $(G^d, \hat{d})$ be the Cauchy completion of G with respect to d. Then the group operations extend to continuous operations on G^d.*

Thus in particular the Cauchy completion will be a topological semi-group.

Fact 0.2 *Let G be a Polish group and let d_1, d_2 be left invariant compatible metrics. Then a sequence in G is Cauchy with respect to d_1 if and only if it is Cauchy with respect to d_2.*

The critical point here is that $(x_n)_n$ will be Cauchy with respect to d_1 if and only if for any open V containing 1 we have

$$x_n^{-1} x_m \in V$$

all sufficiently large n, m. Thus the topological semi-group we obtain by completing with respect to a left invariant metric does not depend on the specific *choice* of the left invariant metric.

Notation Let G be a Polish group. Then $\hat{G}$ is the topological semi-group obtained by completing with respect to any compatible left invariant metric.

Sometimes in the literature $\hat{G}$ is referred to as the completion with respect to the "left uniformity".

Example Let (X, ρ) be a separable complete ultrahomogeneous[1] metric space. let G be the group of all isometries of X *onto* X. Then $\hat{G}$ is (can be identified with) the collection of all isometries of X *into* X.

Here when I say that $\hat{G}$ can be *identified with* I mean that in the topology of point wise convergence and the natural inclusion of $G \hookrightarrow \hat{G}$, we have that $\hat{G}$ is complete with respect to a left invariant metric such as,

$$d(\rho_1, \rho_2) = \sum_{n \in \mathbb{N}} 2^{-n} d'(\rho_1(z_n), \rho_2(z_n)),$$

with $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$ and G is dense.

Example Let U_∞ be the group of linear isometries of ℓ^2, separable infinite dimensional Hilbert space. We give this the topology it obtains from being a closed subgroup of $\mathrm{Isom}(\ell^2)$. Then in fact the completion with respect to the left uniformity of U_∞ equals the collection of all linear isometries from ℓ^2 into ℓ^2.

In general for any Banach space we will similarly obtain that the isometry group is Polish, however its completion with respect to the left uniformity may resist such an easy description. For instance with c_0 and G the group of linear isometries from c_0 onto c_0, one can show that $\hat{G}$ is far smaller than the semi-group of linear isometries from c_0 into c_0.

Example Let $\mathcal{M}$ be a relational ultrahomogeneous[2] structure on $\mathbb{N}$. Let $G = \mathrm{Aut}\mathcal{M}$.

Then $\hat{G}$ consists of all injections $\sigma : \mathcal{M} \to \mathcal{M}$ which preserves quantifier free type.

It turns out that every closed subgroup of S_∞ can be presented as the automorphism group of a countable relational ultrahomogeneous structure. (This is an easy exercise – or see H. Becker, A.S. Kechris, **The descriptive set theory of Polish group actions,** London Mathematical Society Lecture Note Series, 232, Cambridge University Press, Cambridge, 1996 for a proof not only of this result but of various related facts.) Then one can hope to use model theoretic ideas to discuss the Polish groups of this form. In the context of the existence of left invariant metrics one has the following theorem from S. Gao, *On Automorphism Groups of Countable Structures,* **The Journal of Symbolic Logic,** Vol. 63, (1998), pp. 891-896:

---

[1] A metric space is said to *ultrahomogeneous* if for any two finite sequences $\vec{a} = (a_1, a_2, ..., a_n)$, $\vec{b} = (b_1, b_2, ..., b_n)$ with $d(a_i, a_j) = d(b_i, b_j)$ all $i, j \le n$ there is an isometry π of X with $\pi(a_i) = b_i$ at each i

[2] A structure is said to *ultrahomogeneous* if for any two finite sequences $\vec{a} = (a_1, a_2, ..., a_n)$, $\vec{b} = (b_1, b_2, ..., b_n)$ with the same quantifier free type there is an automorphism π of $\mathcal{M}$ with $\pi(a_i) = b_i$ at each i

Theorem 0.3 *$G = \mathrm{Aut}(\mathcal{M})$ has a complete left invariant metric (i.e. $G = \hat{G}$) if and only if there are no uncountable models of its "Scott sentence".*

I do not want to get into a formal definition of Scott sentence here, so let me give an alternate characterization: If $\mathcal{M}$ is countable, and $\mathcal{N}$ is of any cardinality, then $\mathcal{N}$ satisfies the Scott sentence of $\mathcal{M}$ if and only if they satisfy the same sentences in $\mathcal{L}_{\infty,\omega}$, infinitary logic, which by the countability of $\mathcal{M}$ amounts to saying they characterize the same sentences in countably infinitary logic, $\mathcal{L}_{\omega_1,\omega}$. A initially surprising consequence of Gao's theorem is that if a countable model has abelian (and in fact this can be extendd to nilpotent, or solvable) automorphism group, then there are no uncountable models with the same beliefs over $\mathcal{L}_{\omega_1,\omega}$.

The analogy of model theoretic ideas was heavily influential in Howard Becker's proof, from *Polish group actions: dichotomies and generalized elementary embeddings,* **Journal of the American Mathematical Society,** vol. 11 (1998), pp. 397-449, that Polish groups with a complete left invariant metric satisfy Vaught's conjecture.

It also turns out that the completion $\hat{G}$ has been studied extensively in certain schools of mathematicians interested in topological dynamics. Perhaps the best reference I know of for that circle of ideas is V. Pestov's **Dynamics of infinite-dimensional groups. The Ramsey-Dvoretzky-Milman phenomenon.** American Mathematical Society, Providence, RI, 2006. I will give you one example of the interplay with the topological properties of $\hat{G}$ an certain purely combinatorial questions, which I happen to know well since it appeared in G. Hjorth's *An oscillation theorem for groups of isometries,* **Geometry and Functional Analysis,** 18 (2008), 489–521.

Theorem 0.4 *Let G be a Polish group of size bigger than one and let $\hat{G}$ be its completion with respect to the left uniformity. Then there is a bounded, uniformly left continuous*

$$f : \hat{G} \to [0, 1]$$

such for any non-empty right ideal

$$\mathcal{I} \subset \hat{G}$$

we can find

$$\rho_0 \neq \rho_1 \in \mathcal{I}$$

with $f(\rho_0) = 0, f(\rho_1) = 1$.

In the particular case of closed subgroups of S_∞ one can give an independent proof which gives sharper combinatorial information:

Theorem 0.5 *Let $\mathcal{M}$ be an ultrahomogeneous relational structure whose automorphism group has size at least two. Then there is a subset $S \subset \mathcal{M}^2$ on which the automorphism group acts transitively and a function*

$$f : S \to \{0, 1\}$$

such that for any monomorphism

$$\rho : \mathcal{M} \to \mathcal{M}$$

we have a_1, a_2, b_1, b_2 in the image of ρ with

$$(a_1, a_2), (b_1, b_2) \in S,$$
$$f(a_1, a_2) \neq f(b_1, b_2).$$