Math 223D: Topics in Descriptive Set Theory

Time: Mon, Wed, Fri, 1pm-2pm
Place: MSB 5147
Instructor: Greg Hjorth, MSB 7340
Office hours: Tentatively planned for Monday 2pm-3pm, Wednesday 10:30am-12:30pm. In any case, I should be freely available straight after lectures either for shorter questions or to schedule an appointment. You can also grab me after the Logic Seminar on Friday afternoons.
Assessment: For students who want to get an A, there will be three or four homework assignments throughout the quarter.
Reference: There are no required texts for this course, and the lectures should be completely adequate in their own right. However if you want secondary references, then I suggest Kechris' *Classical Descriptive Set Theory* and Becker and Kechris' *The Descriptive Set Theory of Polish Group Actions*.
Content: The aim of this course is to develop the theory of Polish group actions, from the point of view of seeing it as a generalization of countable model theory. Initially we will be begin with introductory material on Polish spaces, Baire category methods, Borel sets, and topological groups. Afterwards we will discuss basic techniques in the study of Polish group actions, as found in Becker-Kechris, such as Birkhoff-Kakutani on the existence of left invariant metrics, Vaught transforms, the Becker-Kechris theorem on changes in topologies, the connections with $\mathcal{L}_{\omega_1,\omega}$, and universal spaces. After having worked through the basics, I want to look at three topics in detail:

(1) The connections between topological groups and countable models. From this point of view, given a countable language $\mathcal{L}$ we can for the space $\mathrm{Mod}(\mathcal{L})$, consisting of all $\mathcal{L}$-structures on $\mathbb{N}$ with the topology generated by first order formulas. There is a corresponding action of the infinite symmetric group S_∞ which induces the isomorphism relation, and one can find a correspondence between elementary model theory and topological properties of the orbits. A sample theorem is that a model is atomic if and only if its orbit is G_δ.

A rather more subtle extension of this to transplant notions of model theory to the general context of arbitrary Polish group actions on Polish spaces. Here in particular we will look at the Vaught transform, which in some sense can be thought of as a generalization of the concept of *type*.

(2) The theory of turbulence This is a rather more recent theory which gives very precise conditions for when a general Polish group action can in some sense be *reduced* to an action of S_∞, and from there the isomorphism relation on a suitable $\mathrm{Mod}(\mathcal{L})$. There are also a number of applications to *non-classifiability* results – to the effect that some notion of equivalence or isomorphism which appears in mathematical practice does not have complete algebraic invariants.

(3) The topological Vaught conjecture The original Vaught conjecture, which Vaught in fact never conjectured, states that "There does not exist a complete first order theory with exactly $\aleph_1$ many countable models up to isomorphism." This has a natural generalization to Polish group actions, in the form of the topological Vaught conjecture. For various classes of Polish groups much stronger results can proved, such as Glimm-Effros type dichotomy theorems. (Some of these use rather extensive metamathematical ideas, and how far we can go in this direction will depend on the audience's familiarity with forcing.)

One another topic: Incidentally, I am thinking of running a participating seminar in some branch of set theory on Wednesday at 5pm. (The day is negotiable, but I cannot do earlier on Monday or Wednesday and Friday is totally out for me.) Basically a participating seminar involves students presenting the material, which is, of course, excellent practice.

If you would be interested, grab me after class or email me at

greg.hjorth@gmail.com