Classification by countable structures

Definition An equivalence relation E on a Polish space X is said to be *classifiable by countable structures* if there is a countable language $\mathcal{L}$ and a Borel function

$$\theta : X \to \mathrm{Mod}(\mathcal{L})$$

such that for all $x_1, x_2 \in X$

$$x_1 E x_2 \Leftrightarrow \theta(x_1) \cong \theta(x_2).$$

Examples: Equivalence relations classifiable by countable structures

(i) Compact zero dimensional spaces up homeomorphism. (A consequence of Stone duality.)

(ii) Discrete spectrum measure preserving transformations. (Implicitly in P. R. Halmos and J. von Neumann, *Operator methods in classical mechanics II,* **Ann. of Math.** 43 (1942), 332–350.)

(iii) Minimal homeomorphisms of the Cantor set up to strong orbit equivalence. (Impicitly in T. Giordano, TI. F. Putnam, C. F. Skau, Christian, *Topological orbit equivalence and C^*-crossed products,* **J. Reine Angew. Math.** 469 (1995), 51–111. .)

(iv) Separable, complete, ultrametric spaces. (Gao and Kechris, see *Polish Metric Spaces: Their Classification and Isometry Groups,* John D. Clemens, Su Gao and Alexander S. Kechris **The Bulletin of Symbolic Logic,** Vol. 7, No. 3 (Sep., 2001), pp. 361-375.)

Examples: Equivalence relations not classifiable by countable structures

(i) Unitary irreducible representations of the free group considered up to isomorphism – i.e. up to unitary conjugation. (More generally any countable group which does not have an abelian subgroup of finite index; unpublished. For groups which are abelian by finite, it follows from the spectral theorem that all their irreducible representations are *finite* dimensional, and hence the responsible unitary groups are compact.)

(ii) Infinite dimensional unitary operators on a separable Hilbert space up to isomorphism. (A.S. Kechris, N. E. Sofronidis, *A strong generic ergodicity property of unitary and self-adjoint operators,* **Ergodic Theory Dynam. Systems** 21 (2001), no. 5, 1459–1479.)

(iii) Separable von Neumann factors. (R. Sasyk, A. Törnquist, *Borel reducibility and classification of von Neumann algebras,* **Bull. Symbolic Logic** 15 (2009), no. 2, 169–183.)

(iv) Measure preserving transformations up to isomorphism – i.e. the conjugation action of $M_\infty(X, \mu)$. (G. Hjorth, *On invariants for measure preserving transformations,* **Fund. Math.** 169 (2001), no. 1, 51–84.)

(v) For Γ a countable, non-amenable group, orbit equivalence of free, mixing actions of Γ. (Epstein, Tsankov).

(vi) The isometry relation of zero-dimensional, complete, separable, metric spaces (Clemens, see *Polish Metric Spaces: Their Classification and Isometry Groups,* John D. Clemens, Su Gao and Alexander S. Kechris **The Bulletin of Symbolic Logic,** Vol. 7, No. 3 (Sep., 2001), pp. 361-375.)