Homework 3, Math 223d, Fall 2009

Due date: Monday, November 30, start of class at 1pm.

1. This is a long extended problem, working towards showing a sharp form of the *Glimm-Effros* (or, if you prefer, the "*Harrington-Kechris- Louveau*") dichotomy for abelian Polish group actions. Some of these steps are harder than others, and so please don't get discouraged if there is some particular part you find more challenging, but move on to the later sections; this is definitely a harder and more involved homework assignment than the last two. Most of the sections do not need the earlier sections to be proved, though at the end one has to put it together to obtain the theorem.

(i) Show that if G is a Polish group and X is a Polish G-space, then there is a Borel function $\theta : X \to 2^{\mathbb{N}}$ such that

$$\theta(x) = \theta(y)$$

if and only if $\overline{[x]_G} = \overline{[y]_G}$. In other words, the equivalence relation of two orbits having the same closures of their orbits is smooth.[1]

(ii) Let X be Polish and let Y be metrizable. Let $\pi : X \to Y$ be a continuous surjection which is also *open* – that is to say that the image of an open set under π is again open in Y. Show that Y is a Polish space.[2]

(ii) Let G be an abelian Polish group and $H < G$ a closed subgroup. Show that G/H in the quotient algebraic and topological structure is a Polish group.[3]

(iii) Let G be a Polish group and X a Polish G-space. Suppose $[x]_G$ is G_δ and for ease of argument assume G acts freely on $[x]_G$ – which amounts to saying that $g \cdot x = x$ if and only if $g = 1$, the identity in G. Show that the evaluation map

$$\phi : G \to [x]_G$$

$$x \mapsto g \cdot x$$

is open.[4]

(iv) Let G be a Polish group and X a Polish G-space. For each $x \in X$ let

$$G_x =_{\mathrm{df}} \{g \in G : g \cdot x = x\}$$

be the *stabilizer* of x. Show that G_x is always a closed subgroup of G.

(v) Let G be an *abelian* Polish group and X a Polish G-space. Show that if $x, y \in G$ with $\overline{[x]_G} = \overline{[y]_G}$, then $G_x = G_y$.

(vi) Let G be an *abelian* Polish group and X a Polish G-space. Let $x \in X$ with $[x]_G$ a G_δ set and assume the action of G on $[x]_G$ is free. Suppose $y \in X$ with $\overline{[y]_G} = \overline{[x]_G}$. Show that xE_Gy – that is to say, they have the same orbit.[5]

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[1] Hint: Let $(U_n)_n$ enumerate a basis for X. Observe that $\{x : [x]_G \cap U_n \neq \emptyset\}$ is Borel (in fact, open!) at each n.

[2] Hint: Separability follows from the topological assumptions, since a dense set in X will have dense image in Y. For completeness, show that Y is G_δ in its Cauchy completion.

[3] **Remark:** In fact this is true even under the assumption of H simply being normal, but in the sequel we will only use it in the case G is abelian. In the abelian case it *might* be a bit easier to see, since we know there will be a complete, compatible, two-sided invariant metric on G.

[4] Hint: This amounts to the inverse map $\phi^{-1} : g \cdot x \mapsto g$ being continuous. It follows from material we looked at in the first couple of weeks of the course that ϕ^{-1} is Borel. Thus it is continuous on a comeager set, C. Then given a sequence $x_n \to x$ in X one can argue that there must exist some $g \in G$ with $g \cdot x_n$ in C at each n and $g \cdot x \in C$. Thus $\phi^{-1}(g \cdot x_n) \to \phi^{-1}(g \cdot x)$.

Remark: In fact one only needs $[x]_G$ is G_δ, and not the assumption that the action is free, to argue for the conclusion of this exercise. But it is a bit easier to see under the assumption the action is free, and this is all we will need in the sequel.

[5] Hint: Let $g_n \cdot x \to y$ and $h_n \cdot y \to x$. Observe that

$$\lim_{n,m\to\infty} h_n g_m \cdot x \to x.$$

(I.e. for each $\epsilon > 0$ there exists finite $F \subset \mathbb{N} \times \mathbb{N}$ such that for all $(n,m) \notin F$

$$d(h_n g_m \cdot x, x) < \epsilon.)$$

Use (iii) to conclude that $(g_n)_n$ is Cauchy in G, and hence for g the limit point, $g \cdot x = y$.

Remark: This time we *really have* used that G is abelian in a serious way. The whole exercise fails without that assumption. For instance, even in the model theoretic case you could probably convince yourself that there exists an atomic model $\mathcal{M} \in \mathrm{Mod}(\mathcal{L})$ of a countable, complete, first order theory in which every element of $\mathcal{M}$ has a different (principal!) type. Then if $\mathcal{N} \in \mathrm{Mod}(\mathcal{L})$ is a non-atomic model of $\mathrm{Th}(\mathcal{M})$ we obtain a counter-example to the exercise in the case of $G = S_\infty$.

(vii) Finally, come home and prove the following surprising theorem:

Theorem: Let G be an abelian Polish group and X a Polish G-space. Then exactly one of the following holds:

(I) E_G is smooth, and in fact xE_Gy if and only if $\overline{[y]_G} = \overline{[x]_G}$;

(II) there is a properly generically ergodic G-space Y and a continuous G-embedding $\pi : Y \to G$.

Remarks: A *G-embedding* is a one to one function with the property that $g \cdot \pi(y) = \pi(g \cdot y)$.

Recall also from lectures that a *properly generically ergodic G-space* is a Polish G-space in which some orbit is dense and every orbit is meager. In fact, the Y in case (II) will be $\{y \in X : \overline{[y]_G} = \overline{[x]_G}\}$ for some $x \in X$ where $[x]_G$ is not G_δ; the computations in (i) show that $\{y \in X : \overline{[y]_G} = \overline{[x]_G}\}$ is G_δ, and hence Polish in the subspace topology.

(II) can be reworked to some extent. From a Becker-Kechris theorem which I discussed in class without fully proving, we can replace (II) by:

(II') $E_0 \leq_B E_G$ – that is to say, there is a Borel function $\theta : 2^{\mathbb{N}} \to X$ such that $z_1 E_0 z_2 \Leftrightarrow \theta(z_1) E_G \theta(z_2)$.

Thus we in particular obtain the conclusion of the Harrington-Kechris-Louveau for orbit equivalence relations induced by abelian Polish group actions. However this theorem is not simply implied by Harrington-Kechris-Louveau, since it was shown in *Equivalence relations induced by actions of Polish groups,* S. Solecki, **Trans. Amer. Math. Soc.** 347 (1995), no. 12, 4765–4777 that abelian Polish groups can often give rise to non-Borel orbit equivalence relations.

On the other hand, I should also point out that Harrington-Kechris-Louveau *cannot* be proved for general Polish group actions: Fix a prime p. Let $\mathcal{L}$ be the language of groups. Let $X \subset \mathrm{Mod}(\mathcal{L})$ be the collection of abelian *p-groups* (for all g there exists $n > 0$ such that $p^n \cdot g = 0$). This is a Polish S_∞-space whose quotient X/E_{S_∞} can be naturally identified with $2^{<\omega_1}$, and it follows from that it is neither smooth nor does it allow a reduction. (This is formally observed in *Analytic equivalence relations and Ulm-type classifications*, G. Hjorth, A.S. Kechris, **J. Symbolic Logic** 60 (1995), no. 4, 1273–1300. This paper also provides a kind of weaker, and essentially optimal, Glimm-Effros dichotomy theorem for general Polish group actions: Either Borel reduces E_0 or in some highly definable manner we can provide elements of $2^{<\omega_1}$ as complete invariants.)