HW2, Math 223d, Fall 2009

Due date: Wednesday, November 11, 1pm at start of class

1. Let X be a Polish space and let $Y \subset X$ be a subset which is Polish in the subspace topology. Show that Y is G_δ in X.

2.Recall that S_∞ is the group of all permutations of the natural numbers with the topology of pointwise convergence. S Show that S_∞ does not have a compatible *complete* left invariant metric. That is to say, there is *no* metric

$$d : S_\infty \times S_\infty \to \mathbb{R}^{\geq 0}$$

such that: (i) it generates the topology on S_∞; (ii) it is complete (i.e. every Cauchy sequence converges); and (iii) $\forall \sigma_1, \sigma_2, \tau \in S_\infty (d(\sigma_1, \sigma_2) = d(\tau\sigma_1, \tau\sigma_2))$.

3. (i) Let X, Y be Polish. Let $B \subset X \times Y$ be Borel. Show that for each open $V \subset Y$

$$\{x \in X : B_x \cap V \,\text{is non} - \text{meager}\}$$

is Borel.[1]

(ii) Let G be a Polish group equipped with a *Borel* action on a Polish space X – in other words, the function

$$G \times X \to X$$

$$(g, x) \mapsto g \cdot x$$

is Borel. At each $B \subset X$ let

$$B^{\Delta V} = \{x \in X : \{g \in V : g \cdot x \in B\} \,\text{is non} - \text{meager}\}$$

is Borel.

(iii) For G, X as above, d a compatible right invariant metric on G bounded by 1, and $B \subset G$ Borel, $V \subset G$ open, $x \in X$, let

$$\varphi_x^{B^{\Delta V}} : G \to [0, 1]$$

by $\varphi_x^{B^{\Delta V}}(g) = \inf\{d(g, h) : h \cdot x \in B^{\Delta V}\}$.

Show that the function

$$X \to \prod_{[0,1]} G$$

$$x \mapsto \varphi_x^{B^{\Delta V}}$$

is Borel.

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[1]Remarks: (a) Here $B_x = \{y \in Y : (x, y) \in B\}$. (b) Probably the natural way to prove this is by some kind of transfinite induction – e.g. show that the sets B with the property we want form a σ-algebra containing the open sets. If you go that route, you may also want to observe at some point in the induction that each set of the form

$$\{x \in X : B_x \cap V \,\text{is co} - \text{meager in} V\}$$

is also Borel.