Math 223d, Fall 2009

Due date: Monday, October 19, 1pm at start of class

1. **(i)** Let X and Y be Polish spaces and let $f : X \to Y$ be a function such that

$$f^{-1}[\mathcal{O}](=_{df} \{x \in X : f(x) \in \mathcal{O}\})$$

is Borel for every open $\mathcal{O} \subset Y$.

Show that f is Borel (i.e. $f^{-1}[\mathcal{B}]$ is Borel for every Borel $\mathcal{B} \subset Y$).

(ii) Let X be Polish and $f : X \to \mathbb{R}$ be such that

$$f^{-1}[(q, \infty)](=_{df} \{x \in X : q < f(x)\})$$

is Borel for each $q \in \mathbb{Q}$.

Show that f is Borel.

2. Let $X = 2^{\mathbb{N}}$, the collection of all functions from $\mathbb{N}$ to $\{0, 1\}$ in the product topology.

(i) Show that for each N and $r \in \mathbb{R}$, the collection $A_{N,r}$ of $f \in X$ with

$$\frac{1}{N}|\{n < N : f(n) = 1\}| > r$$

is Borel.

(ii) Similarly, for each N and $r \in \mathbb{R}$, the collection $B_{N,r}$ of $f \in X$ with

$$\frac{1}{N}|\{n < N : f(n) = 1\}| < r$$

is Borel.

(iii) Show that the set of $f \in X$ such that

$$\text{liminf}_{N\to\infty}\frac{1}{N}|\{n < N : f(n) = 1\}| \geq \frac{1}{2}$$

is Borel. (Hint: $\bigcap_{q<\frac{1}{2}, q\in\mathbb{Q}} \bigcup_{M\in\mathbb{N}} \bigcap_{N\geq M} A_{N,q}$.)

(iv) Similarly for limsup $\leq \frac{1}{2}$.

(v) And thus the set of $f \in X$ with

$$\lim_{N\to\infty}\frac{1}{N}|\{n < N : f(n) = 1\}| = \frac{1}{2}$$

is Borel.