Homework Set Two, Math 223A

April 25, 2001

Saturated Models

1. Show that a countable complete theory has a countable saturated model if and only if it has only countably many (consistent) types.[1]

2. Any two countable saturated models with the same theory are isomorphic.[2]

3. Let c_q be a constant symbol for each $q \in \mathbb{Q}$. Let $\mathcal{M} = (\mathbb{Q}, <, (c_q)_{q\in\mathbb{Q}})$ be the expansion of the rationals obtained by interpreting each c_q to be equal to q. Show that $\mathrm{Th}(\mathcal{M})$ has $2^{\aleph_0}$ many types, and hence has no saturated model.

4. But on the other hand, $\mathrm{Th}(\mathbb{Q}, <)$ (without the resplendant introduction of a cascade of new constant symbols) has a countable saturated model.

5. Does $\mathcal{M} = \mathbb{N}$ with the usual structure (i.e. $(\omega; +, \times, S, 0, 1)$) have a countable saturated model?[3]

6. Let $\mathcal{L}$ be the language generated by unary predicates $(U_n)_{n\in\mathbb{N}}$, and for $\mathcal{M}$ an $\mathcal{L}$-structure and $a \in \mathcal{M}$ we let $f_a : \mathbb{N} \to \{0, 1\}$ be the infinite binary sequence given by

$$f_a(n) = 1 \text{ iff } \mathcal{M} \models U_n(a).$$

Let $S \subset 2^{<\mathbb{N}}$ be a closed downward subset of infinite binary sequences. Let T_S be the theory whose models are exactly $\mathcal{M}$ such that

$$S = \{f_a|_k : k \in \mathbb{N}\}.$$

When does T_S have a countable saturated model?

[1]The "only if" direction is pretty clear. As for the converse direction, show that we may build an elementary chain $\mathcal{M}_0 \prec \mathcal{M}_1 \prec \mathcal{M}_2 \ldots$ such that at each i and for each $\vec{a} \in \mathcal{M}_i$ we have every type consistent over $(\mathcal{M}_i, \vec{a})$ is realized in $\mathcal{M}_{i+1}$.

[2]Note! This is exactly the same result as for atomic models, but the reasoning is rather different.

[3]Hint: Think of the Gödel coding for finite subsets of $\mathbb{N}$.