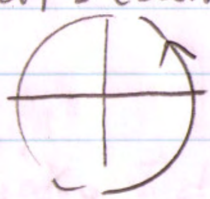
Curves:

a curve is a function $\vec{\gamma}$ from $(a,b) \to \mathbb{R}^2$ or $\mathbb{R}^3$

Ex.: $\mathbb{R}^2$

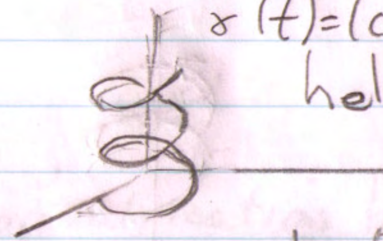
$\vec{\gamma}(t) = (\cos t, \sin t)$



circle

Ex.: $\mathbb{R}^3$

$\vec{\gamma}(t) = (\cos t, \sin t, t)$

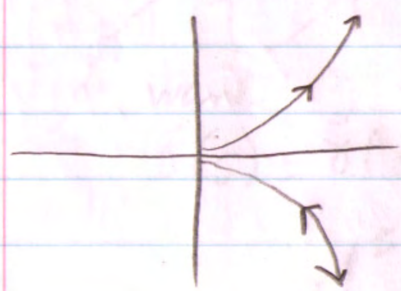


helix

a curve is "smooth" if it is C^k (or C^∞), $k \geq 3$.

C^k means $(x(t), y(t), z(t))$ have k derivatives that are continuous.

Look at this C^∞ curve $\vec{\gamma}(t) = (t^2, t^3)$ $\left(y = \pm x^{3/2}\right)$



problem: it "stops" at 0

$\frac{dx}{dt} = 0$, $\frac{dy}{dt} = 0$, at $t = 0$

We need smooth curves that do not stop

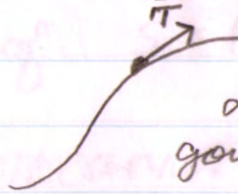
Regular Curves - "Non Singular" Curves

are curves that do not stop

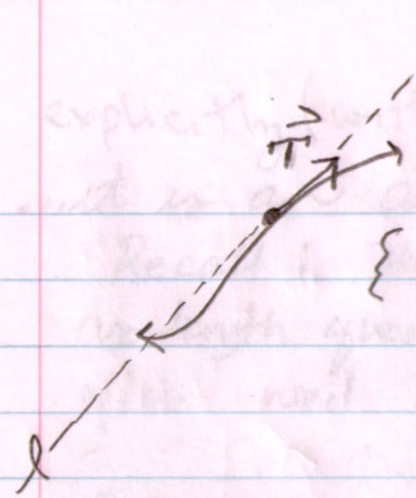
$\vec{\gamma}(t)$: C^3 and $\vec{\gamma}'(t) \neq 0$

The unit tangent at time t is (definition)

$$\frac{\vec{\gamma}'(t)}{\|\vec{\gamma}'(t)\|} = \vec{T}(t)$$



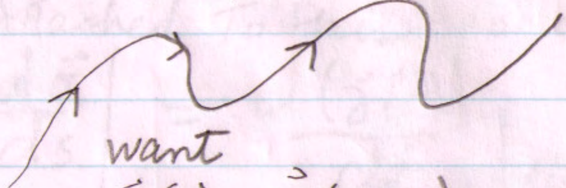
$\vec{T}$ is the direction you are going. Speed is $\|\vec{\gamma}'(t)\|$.



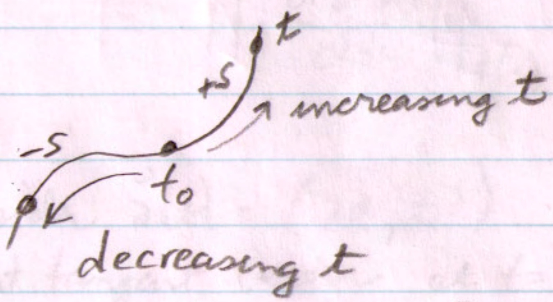
tangent line is
$$\{\vec{\gamma}(t_0) + \lambda \vec{T}(t_0) : \lambda \in \mathbb{R}\}$$

Tricky Idea: Reparametrization by arc length. We have used t (t could be, for example, time) – let's us use s = arc length instead.

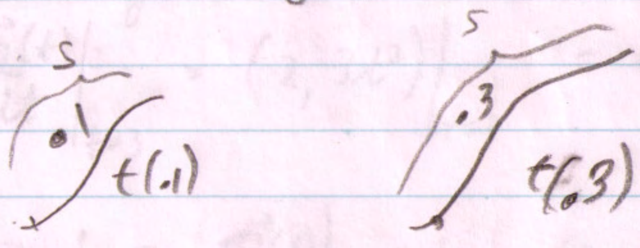
arc length parametrization, $\exists\, t_s \ni \vec{\gamma}(t(s)) = \vec{\sigma}(s)$



want
$$\vec{\sigma}(s) = \vec{\gamma}(t(s)) \quad \text{where} \quad \left\| \frac{d\vec{\sigma}(s)}{ds} \right\| = 1$$



s = arc length along curve



$t(s)$ is the inverse of $s = f(t)$

Most of the time, we cannot find $s = f(t)$ →

explicitly (with a formula)

...it is an arc length function.

Recall: The integrals resulting from arc length questions are "messy" and often need numerical methods. or a CAS.

But the function exists and it has an inverse.

so we have $\vec{\sigma}(s)$, $\left\| \frac{d\vec{\sigma}}{ds} \right\| = 1$

as our arc length parameter.

Attached to this is a unit vector

$$\left.\frac{d\vec{\sigma}}{ds}\right|_s = \frac{\left.\frac{d(\vec{\delta}(t))}{dt}\right|_{t(s)}}{\left\| \frac{d\vec{\delta}(t)}{dt} \right\|_{t(s)}}$$

Our Goal: To understand the geometry of the curve by thinking about how the unit tangent changes along the curve(s).

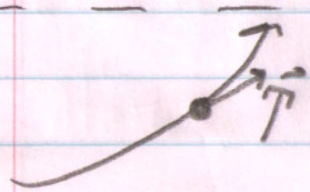
consider $\vec{\delta}(t) = (2t, t^3)$

Unit tangent vector at $t=3$

$$\left.\frac{d\,\vec{\delta}(t)}{dt}\right|_{t=3} = \left.(2, 3t^2)\right|_{t=3} = (2, 27)$$

$\longrightarrow$

and so $\left.\vec{T}(t)\right|_{t=3} = \frac{1}{\sqrt{4+27^2}}(2, 27)$



$\left.\vec{T}\right|_s$ or $\left.\vec{T}\right|_{t(s)}$

The unit tangent tells us in which direction the curve is moving.
If unit tangent vector is constant, the curve is a line.

$\frac{d\vec{T}}{ds} \leftrightarrow$ how does unit tangent change as a function of arc length?

$$= \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} \quad \text{where } \frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

$$\left(\text{or } \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\right)$$

Reparametrization by arc length:

Ex: $\vec{\gamma}(t) = (2t, t^3)$ $\quad \vec{T} = \frac{(2, 3t^2)}{\sqrt{4+9t^4}} = \frac{\frac{d\vec{\gamma}(t)}{dt}}{\left\|\frac{d\vec{\gamma}}{dt}(t)\right\|}$

Now $\frac{ds}{dt} = \sqrt{(2)^2 + (3t^2)^2}$ ☆

$$= \sqrt{4+9t^4}$$

and $\frac{d\vec{T}}{dt} = \frac{d}{dt}\left(\frac{2}{\sqrt{4+9t^4}}, \frac{3t^2}{\sqrt{4+9t^4}}\right)$ ☆☆ ← You do these, please.

to get $\frac{d\vec{T}}{ds}$ use ☆ + ☆☆ .

Claim: $\vec{T}$ and $\frac{d\vec{T}}{ds}$ are $\perp$

Proof:

Let C be the curve that is parametrized by arc length s

$\vec{T}(s)$ is unit tangent $\Rightarrow \langle \vec{T}(s), \vec{T}(s) \rangle = \|\vec{T}(s)\|^2 = 1$

then $\frac{d}{ds} \langle \vec{T}(s), \vec{T}(s) \rangle = 0$

$$\frac{d}{ds} \langle \vec{T}(s), \vec{T}(s) \rangle = \left\langle \frac{d\vec{T}(s)}{ds}, \vec{T}(s) \right\rangle + \left\langle \vec{T}(s), \frac{d\vec{T}(s)}{ds} \right\rangle$$

(Yes, Leibnitz rule works for vector inner products Too)

$$= 2 \left\langle \vec{T}(s), \frac{d\vec{T}(s)}{ds} \right\rangle = 0$$

$\left(\text{since } \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle \right)$

therefore $\langle \vec{T}(s), d\vec{T}(s) \rangle = 0$

and $\vec{T}(s) \perp d\vec{T}(s)$.