Example: Curve C

$$\vec{\gamma}(t) = (2t - t^2, \tfrac{8}{3}t^{3/2}) \quad t \geq 0$$

To Write this w/ arc length parametrization.
We must find a function $\vec{\sigma}(s)$
with $\|\vec{\sigma}'(s)\| = 1$.

We have $s = \int_0^t \|\gamma'(t)\| \, dt = 2\int_0^t t+1 \, dt = t^2 + 2t$

Calculation:

$\gamma'(t) = (2-2t, 4t^{\frac{1}{2}})$

$\|\gamma'(t)\| = \sqrt{4(1-2t+t^2) + 16t}$
$= \sqrt{4t^2 + 8t + 4}$
$= 2\sqrt{t^2 + 2t + 1}$
$= 2\sqrt{(t+1)^2}$
$= 2(t+1)$

$s = t^2 + 2t = f(t)$
$0 = t^2 + 2t - s$
$t = \dfrac{-2 \pm \sqrt{4 - 4(-s)}}{2}$
$= \dfrac{-2 \pm 2\sqrt{1+s}}{2}$

(use +) $t \geq 0$ $\quad t = -1 + \sqrt{1+s}$

$f^{-1}(s) = \sqrt{1+s} - 1$

$s = f(t)$
$f^{-1}(s) = t$

often not possible to find $f(t)$ explicitly, but we could here.

Thus $\vec{\gamma}(t) = \vec{\gamma}(\sqrt{1+s} - 1) =$
$2(\sqrt{1+s} - 1) - (\sqrt{1+s} - 1)^2, \ \tfrac{8}{3}(\sqrt{1+s} - 1)^{3/2}$
$= 4\sqrt{1+s} - 4 - s, \ \tfrac{8}{3}(\sqrt{1+s} - 1)^{3/2}$

This is $\vec{\sigma}(s)$

$\vec{\sigma}(s) = 4\sqrt{1+s} - 4 - s, \ \tfrac{8}{3}(\sqrt{1+s} - 1)^{3/2}$

$$\frac{d\vec{\sigma}}{ds} = \left(\frac{2}{\sqrt{1+s}} - 1, \ \frac{2}{\sqrt{1+s}}(\sqrt{1+s} - 1)^{\frac{1}{2}}\right)$$

and I promise you that

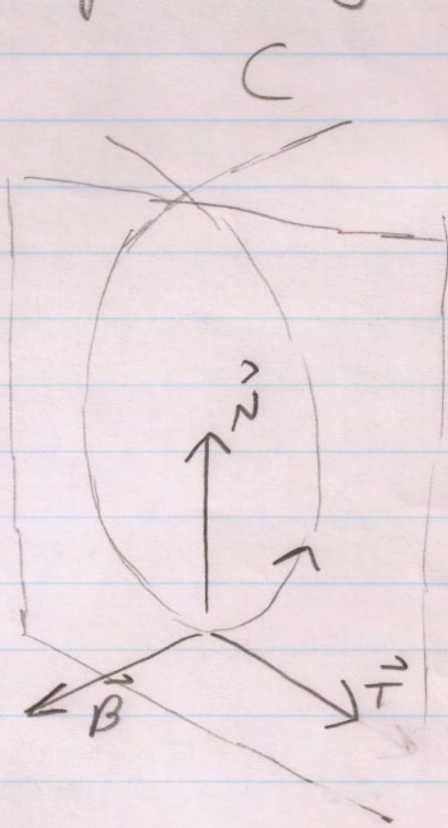
$\left\|\dfrac{d\vec{\sigma}}{ds}\right\| = 1$ (You can check it, it builds character!)

other calculations much more complicated...

②

Back to general situation:

we have $\vec{\sigma}(s)$ arc length parametrization of a curve C



$\left\| \frac{d\vec{\sigma}}{ds} \right\| = 1$

$\vec{T}(s) = \frac{d\vec{\sigma}}{ds}$

$\frac{d\vec{T}}{ds}$ measures change in direction

$\frac{d\vec{T}}{ds} \perp \vec{T}$ why?

Reason:

look at $\langle \vec{T}, \vec{T} \rangle = 1$

$$\frac{d}{ds}\langle \vec{T}, \vec{T} \rangle = 0$$

$$\left\langle \frac{d}{ds}\vec{T}, \vec{T} \right\rangle + \left\langle \vec{T}, \frac{d\vec{T}}{ds} \right\rangle = 0$$

$$2\left\langle \frac{d\vec{T}}{ds}, \vec{T} \right\rangle = 0$$

$$\rightarrow \left\langle \vec{T}, \frac{d\vec{T}}{ds} \right\rangle = 0$$

$$\rightarrow \vec{T} \perp \frac{d\vec{T}}{ds}$$

$\vec{N}$ = unit normal

$\frac{d\vec{T}}{ds} = k\vec{N}$, $\vec{N}$ = unit vector k = curvature $k = \left\| \frac{d\vec{T}}{ds} \right\|$

if $k = 0$ $\vec{N}$ is undefined

and $\vec{B} = \vec{T} \times \vec{N}$ = unit binormal.

This Frame forms an orthonormal basis

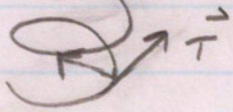
$$\frac{d\vec{T}}{ds} = k\vec{N}$$

$$\frac{d\vec{N}}{ds} = -k\vec{T} + \tau\vec{B}$$

$$\frac{d\vec{B}}{ds} = -\tau\vec{N}$$

Example C: $\vec{\gamma}(t) = (\cos t, \sin t, t)$

Here is a spiral



$\gamma'(t) = (-\sin t, \cos t, 1)$

$|\gamma'(t)| = \sqrt{2}$

$s = \int_0^t \sqrt{2}\, dt = \sqrt{2}\, t \qquad \frac{s}{\sqrt{2}} = t$

$\vec{\gamma}(s) = \left(\cos \frac{s}{\sqrt{2}}, \sin\frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}\right) \to$ arc length parametrization.

$\vec{T}(s) = \left(-\frac{1}{\sqrt{2}} \sin\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$\frac{d\vec{T}}{ds} = \left(-\frac{1}{2}\cos\frac{s}{\sqrt{2}}, -\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right) = +\frac{1}{2}\vec{N}$

$k = \left\|\frac{d\vec{T}}{ds}\right\| = \frac{1}{2}$

$\vec{N} = \left(-\cos\frac{s}{\sqrt{2}}, -\sin\frac{s}{\sqrt{2}}, 0\right)$

$\vec{T} \times \vec{N}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\cos\frac{s}{\sqrt{2}} & -\sin\frac{s}{\sqrt{2}} & 0 \end{vmatrix}$$

$\left(+\frac{1}{\sqrt{2}}\sin\frac{s}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\cos\frac{s}{\sqrt{2}}, +\frac{1}{\sqrt{2}}\right) = \vec{B}$

$\frac{d\vec{B}}{ds} = \left(+\frac{1}{2}\cos\frac{s}{\sqrt{2}}, +\frac{1}{2}\sin\frac{s}{\sqrt{2}}, 0\right) = -\frac{1}{2}\vec{N}$

$\tau = 1/2 \qquad = -\tau\vec{N}$

Example:

Circle radius R —

Find arc length parametrization —

$$\vec{\gamma}(t) = (R\cos t, R\sin t, 0)$$

$$\vec{\gamma}'(t) = (-R\sin t, R\cos t)$$

$$\|\vec{\gamma}'(t)\| = R$$

$$s = \int_0^t R\,dt = Rt$$

$$s = Rt \rightarrow \frac{s}{R} = t$$

$$\vec{\sigma}(s) = \vec{\gamma}\left(\frac{s}{R}\right) = \left(R\cos\left(\frac{s}{R}\right), R\sin\left(\frac{s}{R}\right), 0\right)$$

and notice that

$$\|\vec{\sigma}'(s)\| = 1$$

$$\vec{T}(s) = \left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right), 0\right), \quad \|\vec{T}(s)\| = 1$$

$$\frac{d\vec{T}}{ds} = \left(-\frac{1}{R}\cos\left(\frac{s}{R}\right), -\frac{1}{R}\sin\left(\frac{s}{R}\right), 0\right)$$

$$= \frac{1}{R}\left(-\cos\left(\frac{s}{R}\right), -\sin\left(\frac{s}{R}\right), 0\right)$$

$$\frac{d\vec{T}}{ds} = k(s)\vec{N}$$

$$\rightarrow k(s) = +\frac{1}{R}$$

$$k(s) = \|T'(s)\| = \frac{1}{R}$$

$$\frac{d\vec{T}}{ds} = \frac{1}{R}\left(-\cos\left(\frac{s}{R}\right), -\sin\left(\frac{s}{R}\right), 0\right)$$

$$\vec{N} = \left(-\cos\left(\frac{s}{R}\right), -\sin\left(\frac{s}{R}\right), 0\right)$$

Claim: The Torsion is 0. Reason:

$\rightarrow$

$\vec{T} \times \vec{N} = \vec{B}$, the unit binormal.

$\vec{N}$ and $\frac{d\vec{N}}{ds}$ are $\perp$ $\rightarrow$ $\frac{d\vec{N}}{ds}$ is a linear combination of $\vec{T}$ and $\vec{B}$

$$\frac{d\vec{N}}{ds} = \left(\frac{1}{R}\sin\left(\frac{s}{R}\right), -\frac{1}{R}\cos\left(\frac{s}{R}\right), 0\right)$$

$$= -\frac{1}{R}\left(-\sin\left(\frac{s}{R}\right), \cos\left(\frac{s}{R}\right), 0\right) = -k\vec{T}$$

but $\frac{dN}{ds} = -k\vec{T} + \tau\vec{B}$ $\Rightarrow \tau = 0$

Fact: all plane curves have Torsion 0