String theory and balanced metrics

One of the main motivations for considering balanced metrics, in addition to the considerations already mentioned, has to do with the theory of what are known as heterotic strings with super-symmetry. The physics here is complicated to explain, having to do with accommodating both bosons and fermions simultaneously. But the mathematics involved has a reasonably straight-forward expression, though it is far from obvious how to find interesting examples of the type of system involved. To jump immediately to the description of the structure one would like to find(we shall return later to motivational matters) we would like to find a complex manifold M of complex dimension 3, a vector bundle $V \to M$ with a Hermitian Yang Mills metric h and associated Hermitian type (1,0) connection with curvature F_h (as in our previous notation) and a holonorphic 3-form on M, denoted by Ω, which satisfy several equations:

1 $F_h^{2,0} = F_h^{2,0} = 0$ (this is equivalent to the bundle being holomorphic and the connection being Hermitian)

2 $F_h \wedge \omega^2 = 0$ (equivalently, $c_1(V)=0$)

3 $d\|\Omega\|_\omega \omega^2 = 0$ where $\|\Omega_\omega\| =$ by definition $\Omega \wedge \overline{\Omega} / \omega^3$.

(In previous cases, where one had Ricci flatness, one was looking for an Ω with constant norm in condition 3, so that the equation became just that of a balanced metric.)

To these, one wants to add a fourth condition that arises from physics, what is known as the anomaly cancellation, namely that

4 $\sqrt{-1}\partial\overline{\partial}\omega = trR_\omega \wedge R_\omega - trF_h \wedge F_h$

Solving equations 1 and 2 amounts just to the Hermitian Yang Mills situation that we have discussed before. Namely, it can be done is ω is a balanced metric and V is holomorphic and stable with respect to ω.

In order to deal with equation 3, leaving aside for the time being condition 4, one could think of introducing a new two-form $\tilde{\omega} = \|\Omega\|^{1/2} \omega$. Then $\tilde{\omega}$ has $d\tilde{\omega}^2 = 0$. So one can investigate first just the condition that for some (1,1) form ω, $d\omega^2 = 0$. There are actually several constructions that can be used here. We consider one that involved branch covers.

Suppose $L\to M$ is a line bundle with a nontrivial holomorphic section s and let D be the divisor determined by s=0. And suppose further that one has for some reason or another that $c_1(L)$ is divisible by a positive integer m >1 (in the Z cohomology, that is the quotient is still an integral class) . Then for some line bundle L' , L is the mth power of L'. This corresponds to a map on the fibres $(x,z)\to(x, z^m)$. [Note that if L has a topological mth root in this sense, then it has a holomorphic mth root, since the only obstruction to taking the mth roots of the transition functions coherently is topological. Thus the divisibility of the first Chern class of L is the only obstruction.] Let $F:L'\to L$ be this mth power fibre map. Then in the total space of L', one can consider F^{-1} (s(M)) , where s(M) denotes the image under s in the total space of L of the base manifold M. Here we are interpreting s as a map $M\to L$ in the usual way. The set F^{-1} (s(M)) is a branched cover over M with m "sheets" (m preimages of a generic point in M) and m-fold branching over the zeroes of the section s.

The importance of this construction for us is that the branch-cover manifold admits a balanced metric(assuming that M does). For this, note that we can regard the branch cover as a submanifold of LxL' by sending a point in the branch cover (x,s), s in the fibre of L' over x to (x, s, s^m) in LxL' over M. We put a metric on this submanifold as follows: write for each open set in a trivializing cover U_α, $\|z\|^2 = h_\alpha(|z|^2)$. Considering $\|z\|^2$ as a function of all variables, we can consider $\partial\bar{\partial}\|z\|^2$. This may not be positive but is positive in fibre directions, if perhaps not in base manifold directions. (In the Calabi Conjecture negative curvature case, this is actually positive in base directions also and is a positive definite metric in all variable, in fact.) Now consider the sum , for some (large positive) constant C :

$(\partial\bar{\partial}\|z\|^2)^{n-1} + C\pi^*\omega^{n-1}$ where π is the projection onto M.

Since we are working on a compact set(a compact submanifold, the image of M), we can choose C so large that this is in fact a metric on the subvariety. And similarly on LxL', we can use

$(\partial\bar{\partial}\|z'\|_{L'}{}^2)^{n-1} + (\partial\bar{\partial}\|z\|^2)^{n-1} + C\pi^*\omega^{n-1}$ to get a positive form on the (compact) submanifold of LxL' already discussed (which is indeed the branched cover). Note here that the concept of positivity of a (p,p) form makes sense without respect to the overall dimension. So we can restrict to F^{-1}(s(M)) , and noting that all forms are closed, we get a balanced metric on the branch cover F^{-1}(s(M)) .

Note that in fact we constructed not the Kahler form itself of the balanced metric but rather a positive closed (n-1,n-1) form (on the n dimensional branch cover) directly. But as noted earlier, every such (n-1, n-1) form arises as the (n-1) st wedge product power of a uniquely determined positive (1,1) form, point by point, by linear algebra(cf the earlier discussion of balanced metrics).

In this situation if D is nonsingular then the branched covering is also nonsingular. Unfortunately, it often arises that one wishes to deal with singularities in the divisor D and one then needs some resolution work. This should usually be doable, however.

Now let us return to the situation of twistor space Tw(M) for a conformally self-dual four manifold M , as discussed earlier. In this case, one has a fibration $CP^1 \to Tw(M) \to M$, Tw(M) is a complex manifold(from the self duality) and Tw(M) is balanced in its natural metric as discussed earlier also.

However, in general $c_1(M)$ is nonzero so that one cannot find nontrivial holomorphic 3-forms without zeroes. (Recall that we wanted to use the norm of a holomorphic form as a multiplication factor to get a new metric so one needs to have a holomorphic form and have that the form is nowhere 0, too.)

To deal with this, we might try to find a divisor in twistor space that would function to give a branched cover with respect to which the construction can be carried out. In short, we hope to "kill" (trivialize) the canonical bundle by a branched cover.

This is related to an idea in Riemann surface theory. By forming suitable branch covers, one can introduce holomorphic forms where none existed before.

Recall the usual form of this idea:

Suppose N is a compact Riemann surface and let K^{-1} be the anti-canonical bundle (of local sections of the holomorphic tangent bundle. In higher dimensions this would be the n times wedge product of the holomorphic tangent space, n= complex dimension). Then if s is a meromorphic section of K^{-1}, then it has locally the form $f(z)(\partial/\partial z)$ where f is a meromorphic function(similar in higher dimensions). So s^{-1} has a pole along the divisor D, the anticanonical divisor defined by the vanishing locus of s, the section of the anit-canonical bundle. We want to get rid of the pole. And the thing to do is to pass to a branch cover over the poles of s^{-1}.

But even for Riemann surfaces, this situation is more subtle than at first appears. Imagine a branch covering of order m, say defined in local coordinates by $u^m = z$, where z is the base, u is in the cover. Then the coordinate vector field $\partial / \partial u$ pushed down (at a given point where u is not 0) becomes a multiple of $\partial / \partial z$ by the complex number mu^{m-1}. This goes to 0 as u goes to 0 but it has "fractional order" as a function of (the magnitude) of z, namely it has order of magnitude $|z|^{1-(1/m)}$. So one picks up some advantage but only in terms of fractional order. In bundle terms, one is really gaining only in a bundle a power of which is the given anti-canonical bundle.

This somewhat 19th century description actually works. Think of the anti-canonical bundle of CP^1. A holomorphic section of this(and there are some) has total degree 2, that is the sum of the orders of the zeroes is 2. If one is looking at two -fold branch covers, then according to the above "heurisitic", each (two -fold) branch point account for a gain of 1-(1/2)= ½. So one expects to need four such branch points to get a branched cover with a holomorphic nowhere zero (1,0) form. This is the right answer! By the Riemann – Hurwitz formula, such a cover has Euler characteristic 2 x2 – 4x1=0, where the terms are respectively 2 (Euler characteristic of the base) and sum of branching order -1 over all branch points. So the branched cover is a torus and the required holomorphic (1,0) form exists.

Similarly , a triple (three-sheet) covering with three triple branch point will give a branched cover with Euler characteristic 0 and thus with a trivial canonical bundle. Here in the heuristic above each branch gives an advantage of 1-(1/3) so that three of such branch points gets rid of 3(2/3) of the total order of zero 2 for the holomorphic tangent bundle section, and again one sees that one is in the trivial canonical bundle situation in the branched cover.

Clearly, this is a quite subtle matter to deal with in general cases. But the general principle applies that one gets advantage passing to branched covers but in fractional amounts depending on the branching order, the fractions corresponding in effect to "roots" (locally) of the bundle.

In higher dimensions, an interesting set of examples are the K3 surfaces. These Kahler surfaces all admit Ricci-flat metrics by Yau's solution of the Calabi Conjecture in case of first Chern class 0. These manifolds are anti-self-dual automatically with respect to this metric(a Kahler metric is anti-self-dual if and only if its scalar curvature vanishes).
By the Bochner technique argument discussed earlier, any holomorphic (2,0) form in this case is actually parallel with respect to the Kahler –metric connection. And such a form Ω can be chosen to have $\Omega \wedge \overline{\Omega} = \omega^2$. This shows the important observation that the holonomy reduces from U(2) to SU(2) ! since the form Ω is parallel from which one sees that the holonomy action on the volume form is not just of absolute value 1(as is inevitable from parallel translation being an isometry) but is actually equal exactly to 1.

Now in this case, one can find additional complex structures as follows:
The choice of $T^{0,1}$ determines an almost complex structure. For a form A not zero at a point and of type (2,0), one has that $T^{0,1}$ is determined via $\{X: i_X A=0\}$. Now we are in possession of three two-forms, namely $\Omega, \overline{\Omega},$ and ω. So we can form new almost complex structures(which are actually integrable) by chosing various possibilities for A.

Namely, we can build in this situation a family of integrable almost complex structures by taking $A= a\Omega + b\omega + c\overline{\Omega}$ where $a^2 + b^2 + c^2 = 1$, a,b,c, constants. This is still closed, and because all three are parallel, this form is never 0 if it is nonzero at one point. This gives a new integrable complex structure so one has a sphere of integrable almost structures, corresponding to the sphere fibre at one point in the twistor space over M, translated to every point by parallelism. And again by parallelism and the construction, all these are compatible with the metric g.

Note that in this situation, one has that Tw(M) is smoothly(not biholomorphically) the product of M and CP^1, where the coordinates a,b,c are the coordinates on CP^1 identified as usual with the unit sphere in R^3.

Now there are three sections here ,thinking of Tw(M) as a complex bundle over M with CP^1 fibres. So as we have been discussing, we can form a branch covering over the divisors corresponding to the images of these sections in Tw(M). This produces an elliptic fibre space over the K3 surface M: the fibres are elliptic curves that arise from the branching over the three points in each CP^1 fibre over a point in M . (As we observed earlier , a three-sheeted cover of CP^1 branched over three points, each with branching order 3 , is a torus as a Riemann surface, that is, an elliptic curve). This new space, the elliptic curve fibration over the K3 surface admits a holomorphic 3-form. Thus one obtains a large class of compact complex manifolds of dimension 3 with $c_1=0$ and with a holomorphic 3-form. This is an interesting situation to consider.

There are other interesting ideas for constructing examples of the type we are interested in beyond the idea we have been discussing , of branched coverings arising from twistor spaces. One of these is a generalization of the Calabi-Eckmann construction.
Recall that in the classical Calabi-Eckmann construction, one considers, for example, the fibration $S^1 \times S^1 \rightarrow S^3 \times S^3 \rightarrow CP^1 \times CP^1$ and the complex structure on the product of the 3-spheres is built from that on the product of the projective spaces and the complex structure on the torus fibres. Now Goldstein and some physicists suggested that one could look at the following generalization:

Choose two line bundles(which could be the same one) L_1 , L_2 over a K3 surface M, or an arbitrary Calabi-Yau manifold. We want to have here that the (first) Chern form $c_1(L_i) \wedge \omega = 0$ so as to have an anti-self-dual Chern form. There is in fact a large set of line bundles for which this will happen. This follows from dimension considerations. The (complex coefficient) cohomology of the K3 surface is dimension (over C) = 22 . Only three of these dimensions are generated by the Kahler form and the holomorphic (2,0) parallel form and its complex conjugate. The rest of these are generated by forms which are type (1,1) and have wedge product with ω=0.

Now take the principal circle bundles associated to two line bundles(possibly equal) of the sort indicated. Then the torus bundle over M attached to this situation has a natural complex structure(details of this will be covered later).

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