Lecture 6 Part I
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Allowable singularities and Compactifications

To treat the class VII surfaces with $b_1=1$ and $b_2=0$ but with curves on the surface, one needs to think about extending the idea of sections of bundles to sections with specified polar sets (along the curves) and metrics with singular sets of a certain type.

The basic idea comes from thinking about the metric of constant negative Gauss curvature on a compact Riemann surface with punctures (other than CP^1 if there is only one puncture). One can compute easily enough what the expected form of the metric is near a puncture. In a coordinate system where the puncture is at z=0, it should look like
$(1/ |z| \cdot \ln |z|)^2 (dx^2 + dy^2)$. [This form arises from noting that the upper half plane covers by $f(z) = \exp(iz)$ the unit disc with 0 removed. The push-down of the Poincare metric on the upper half plane is a rotationally symmetric metric on the punctured unit disc and , using the fact that the Poincare metric on the upper half plane is $(1/y^2)(dx^2 + dy^2)$, one computes directly the form of the metric as given. Here one notes that the y of the preimage of a point of absolute value r is –log r , from which the log z term in the metric arises.]

The Chern form for the holomorphic tangent bundle
[which is $-(1/2\pi)$(Gauss curvature)$\bullet$ (oriented area form), in the Riemann surface case] is easily computed to have finite integral in this case: in polar coordinates one is integrating , down to 0+, the function $(1/r)(1/\ln r)^2$ which has finite integral. Looking at this carefully, one sees that in fact the integral over the punctured surface of the first Chern form is the first Chern class of the line bundle corresponding to holomorphic (1,0) forms with at most simple poles at the punctures. This is checked by a direct examination of the limiting behavior of integral at the puncture showing that each puncture adds 1 to what would be the (negative of)Euler characteristic integral , while the degree of the bundle as described is simply the degree of the canonical bundle plus the number of punctures.

This whole observation can be transferred to higher dimensions, with the distinguished points , the punctures as it were, replaced by curves or more generally divisors with normal crossings.

For instance, for curves on a surface, if a curve is defined locally by $z_1=0$ then one allows differentials of the form $f\,(dz_1 / z_1) \wedge dz_2$. In general, one can look at this from the sheaf viewpoint and include, in the normal crossing case of a curve defined by $z_1z_2= 0$, singularities for the differentials of the form $f\,(dz_1 /z_1) \wedge (dz_2 /z_2)$. This can be extended to higher dimensions as well.

One considers again a divisor with normal crossings and looks at singular metrics with the same Poincare-type singularities as described, in other words sums of terms of the form $[1/ \left|z_i\right|^2 (\ln \left|z_i\right|)^2]\ (dz_i)^2$ for each i from 1 to k if the normal crossing of the divisor is defined by $z_1 \ldots z_k = 0$. An Hermitian metric on a line bundle on the complement of the divisor in the manifold is called "good" (Mumford's terminology) if the curvature F_h arising from the h-determined connection on L does not grow faster than the Poincare type of growth indicated. In this case, one can define integration of tr $(F_h \wedge \ \ldots\ldots \ \wedge F_h)$
as a closed current in M. Then the Chern forms defined as before represent the Chern classes of the extension of the bundle to allow singularities of the divisor type, in exact analogy to the situation of divisors on Riemann surfaces.

This turns out to work in the situation of Hermitian symmetric spaces. One takes an Hermitian symmetric domain D(e.g. the Siegel upper half space of matrices of the form X+iY where X and Y are symmetric and Y is positive definite). Then one considers a quotient of the form $M= D /\Gamma$, where Γ is a discrete group of holomorphic transformations of D. The quotient may be compact or not, but if noncompact is usually required to be of finite volume. Then one forms compactifications of this quotient item.

What compactification means in this case is more specific than just topological compactification. We require that $\overline{M}$ is a complex variety and $\overline{M}$ –M is a subvariety of $\overline{M}$, so M is a "Zariski open set in the compactification $\overline{M}$.

There are a number of such compactification, Bailley-Borel, Satake…But especially important for our purposes is the Mumford "toroidal compactification" . This is important because in this case $\overline{M}$ is smooth and $\overline{M}$ –M has normal crossings. Mumford showed that all natural bundles satisfy his "goodness" condition with natural metrics. ("Goodness" here of course depends on the compactification actually).

To understand class VII_0, case 2, one could try to look at a similar idea. The examples known in fact look like HxC /Γ, quotient by a discrete group. (recall that Γ acts by affine linear transformations in this case). The group does not preserve a metric –it is not quite like an Hermitian symmetric space case. But it does preserve the affine structure.

It should be possible (one hopes: not done yet) to do compactification to get a situation where there is a complete metric outside a finite union of curves which is projectively flat on the bundle-with-singularities-allowed considered earlier.

Ideas for complex dimension 3

First question: Which manifolds have a complex structure with a Kahler form ω having $\partial\bar{\partial}\omega^{n-1}$ =0?

Note that this does not always happen!

Last time we saw that the "$\partial\bar{\partial}$" Lemma gave a condition in principle intermediate on a compact manifold between being a Kahler manifold(a complex manifold admitting a Kahler metric) and just being a complex manifold. Here is another such condition: that a holomorphic (1,0) form is always closed. [On a compact Kahler manifold, a holomorphic form α of type (p,0) is always closed, for any p>0. To see this, consider the Hodge decomposition of α relative to $\bar{\partial}$ harmonic theory. Since α is $\bar{\partial}$ closed,it must be that α is the sum of a form in the image of $\bar{\partial}$ and a $\bar{\partial}$-harmonic form. But by types, the image of $\bar{\partial}$ part is 0. So α is $\bar{\partial}$ -harmonic and hence d-harmonic and hence closed. This is a global fact of course: f dz_1, f a locally defined nonconstant holomorphic function, z_1 a local coordinate function, is a holomorphic (1,0) form which is not closed.]
This is actually implied by the "balanced" condition we defined earlier, that $\partial\bar{\partial}\omega^{n-1}$ =0. This is proved by integration by parts.

[In detail, recall that we showed earlier that $\int \partial\sigma \wedge \tau$ =0 if and only if

$\int \sigma \wedge \partial\tau$ =0 and similarly that $\bar{\partial}$ (and of course d itself) can be "moved to

the other side". This was a simple consequence of Stokes' Theorem; the two integrals are

in fact always equal up to a +- sign. Now consider, for a balanced metric and a (1,0)

holomorphic form θ, $\int \partial\theta \wedge \overline{\partial\theta} \wedge \omega^{n-1}$.

Ignoring irrelevant +- signs, we have

$\int \partial\theta \wedge \bar{\partial}\bar{\theta} \wedge \omega^{n-1} = \int \theta \wedge \partial(\bar{\partial}\bar{\theta} \wedge \omega^{n-1}) = \int \theta \wedge \partial\bar{\partial}\bar{\theta} \wedge \omega^{n-1} + \int \theta \wedge \bar{\partial}\bar{\theta} \wedge \partial\omega^{n-1} =$ (since $\partial\bar{\partial}\theta$=0)

$\int \theta \wedge \bar{\partial}\bar{\theta} \wedge \partial\omega^{n-1} = \int \bar{\partial}\theta \wedge \bar{\theta} \wedge \partial\omega^{n-1} + \int \theta \wedge \bar{\theta} \wedge \partial\bar{\partial}\omega^{n-1}$ =0

since $\bar{\partial}\theta$=0 and $\partial\bar{\partial}\omega^{n-1}$=0.

So if $\partial\bar{\partial}\omega^{n-1}$ is 0, it follows that $\int \partial\theta \wedge \bar{\partial}\bar{\theta} \wedge \omega^{n-1}$=0 if θ is holomorphic, and hence that under these conditions θ is closed, as required.]

Thus we have a condition somewhere between just being a complex manifold and being a Kahler manifold, namely that all holomorphic (1,0) forms are closed.