Math 121 Homework II

1.(a) First note that any rational ball \( B(p, r) \) is uniquely determined by the point \( p \in \mathbb{Q}^n \) and \( r \in \mathbb{Q}_{>0} \) (this means positive rationals). Hence we have an injection into \( \mathbb{Q}^{n+1} \). So it suffices to show that \( \mathbb{Q}^n \) is countable for any \( n \geq 1 \). This is true for \( n = 1 \). Assume true for some \( n \geq 1 \). Now for any \( q \in \mathbb{Q} \), the set \( \{ (q, p) : p \in \mathbb{Q}^n \} \) is in bijection with \( \mathbb{Q}^n \), hence countable by assumption. Then

\[
\mathbb{Q}^n = \bigcup_{q \in \mathbb{Q}} \{ (q, p) : p \in \mathbb{Q}^n \}
\]

is a countable union of countable sets, hence countable. By induction the result follows.

1.(b) Let \( B_U = \{ (p, r) : p \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0} \text{ and } B(p, r) \subseteq U \} \). Since each \( B(p, r) \in B_U \) is contained in \( U \) it is clear that

\[
\bigcup_{B(p, r) \in B_U} B(p, r) \subseteq U.
\]

Now take any \( x \in U \). Since \( U \) is open there is some \( \varepsilon > 0 \) such that \( B(x, \varepsilon) \subseteq U \). Be density of the rationals we know there is some \( p \in \mathbb{Q}^n \) such that \( |x - y| < \varepsilon/3 \). By density of the rationals, again, we know that there is some \( r \in \mathbb{Q} \) with \( \varepsilon/3 < r < \varepsilon/2 \). Then note that \( x \in B(p, r) \) and for any \( y \in B(p, r) \) we have

\[
\|y - x\| \leq \|y - p\| + \|p - x\| < \varepsilon/2 + \varepsilon/3 < \varepsilon,
\]

so \( B(p, r) \subseteq U \). It follows that

\[
U \subseteq \bigcup_{B(p, r) \in B_U} B(p, r).
\]

2. For each \( \lambda \in \Lambda \) let \( B_\lambda = \{ (p, r) : p \in \mathbb{Q}^n, r \in \mathbb{Q}_{>0} \text{ and } B(p, r) \subseteq U_\lambda \} \) and let \( \mathcal{B} = \bigcup_{\lambda \in \Lambda} B_\lambda \). By question 1.(a) we know that \( U_\lambda = \bigcup_{B(p, r) \in B_\lambda} B(p, r) \), hence

\[
S \subseteq \bigcup_{\lambda \in \Lambda} U_\lambda = \bigcup_{\lambda \in \Lambda} \left( \bigcup_{B(p, r) \in B_\lambda} B(p, r) \right) = \bigcup_{B(p, r) \in \mathcal{B}} B(p, r).
\]

By question 1.(a) we see that \( \mathcal{B} \) is countable (being a subset of a countable set). Write \( \mathcal{B} = \{ (p_i, r_i) : i \geq 1 \} \). For each \( i \geq 1 \) take \( \lambda_i \in \Lambda \) such that \( B(p_i, r_i) \subseteq U_{\lambda_i} \), i.e. \( B(p_i, r_i) \subseteq \bigcup_{i=1}^\infty U_{\lambda_i} \). Then

\[
S \subseteq \bigcup_{B(p, r) \in \mathcal{B}} B(p, r) = \bigcup_{i=1}^\infty B(p_i, r_i) \subseteq \bigcup_{i=1}^\infty U_{\lambda_i}.
\]

3. Assume \( S \) does not contain any condensation point of itself. Then for every \( x \in S \) there is some \( \varepsilon_x > 0 \) such that \( B(x, \varepsilon_x) \cap S \) is at most countable. Now

\[
S \subseteq \bigcup_{x \in S} B(x, \varepsilon_x)
\]

and by question 2 we can find some countable subcollection \( \{ x_i : i \geq 1 \} \) such that

\[
S \subseteq \bigcup_{i=1}^\infty B(x_i, \varepsilon_{x_i}).
\]

Then

\[
S \subseteq \left( \bigcup_{i=1}^\infty B(x_i, \varepsilon_{x_i}) \right) \cap S = \bigcup_{i=1}^\infty (B(x_i, \varepsilon_i) \cap S),
\]

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which is a countable union of at most countable sets, hence countable.

4.(a) Let $C \subseteq \mathbb{R}^n$ be the set of condensation points of $S$ and take $x \in \mathbb{R}^n \smallsetminus C$. Since $x$ is not a condensation point of $S$ there is some $\varepsilon > 0$ such that $B(x, \varepsilon) \cap S$ is at most countable. Take $y \in B(x, \varepsilon)$ and set $\delta = \varepsilon - ||x - y||$. Then we have that $B(y, \delta) \subseteq B(x, \varepsilon)$ and $B(y, \delta) \cap S \subseteq B(x, \varepsilon) \cap S$. Thus $B(y, \delta) \cap S$ is at most countable, so $y \in \mathbb{R}^n \smallsetminus C$, that is $B(x, \varepsilon) \subseteq \mathbb{R}^n \smallsetminus C$. It follows that $C$ is closed.

4.(b) Let $C$ denote the condensation points of $S$ and let $p \in C$. For each $n \geq 1$ we see that $B(p, 1/n) \cap S$ is uncountable, hence $(B(p, 1/n) \cap S) \smallsetminus \{p\}$ is uncountable. By question 3 there exists some $x_n \in B(p, 1/n)$ with $x_n \neq p$ and $x_n$ a condensation point of $S$. Since $||x_n - p|| < 1/n$ for all $n \geq 1$ we see that the sequence $(x_n)_{n \geq 1}$ converges to $p$.

5.(a) Let $C$ denote the set of condensation points of $S$. Assume that $S \smallsetminus C$ is uncountable. Then, by question 3, $S \smallsetminus C$ has a condensation point belonging to $S \smallsetminus C$, i.e. there is some $p \in S \smallsetminus C$ such that for any $\varepsilon > 0$, $B(p, \varepsilon) \cap (S \smallsetminus C)$ is uncountable. Hence, for every $\varepsilon > 0$, $B(p, \varepsilon) \cap S$ is uncountable and $p$ is a condensation point of $S$. This is a contradiction.

5.(b) Let $C$ denote the set of condensation points of $S$. By the definition of condensation points of $S$ we see that any condensation point of $S$ is adherent to $S$. Since $S$ is closed we conclude $C \subseteq S$, i.e. there is no $p \in S \smallsetminus C$ such that for any $\varepsilon > 0$, $B(p, \varepsilon) \cap (S \smallsetminus C)$ is uncountable. Hence, for every $\varepsilon > 0$, $B(p, \varepsilon) \cap S$ is uncountable and $p$ is a condensation point of $S$. This follows from 4.(a).

6. Let $C$ denote the Cantor set. Then $C = \cap_{i=0}^{\infty} E_n$ where each $E_n$ is a union of $2^n$ closed intervals each of length $3^{-n}$ constructed inductively as follows. Let $E_0 = [0,1]$. Assuming $E_n$ is constructed we construct $E_{n+1}$ by deleting from $E_n$ the open interval $(a_i + 3^{-(n+1)}, b_i - 3^{-(n+1)})$ from each of the $2^n$ closed intervals $[a_i, b_i]$ in $E_n$. Note that since each $E_n$ is a finite union of closed intervals it is closed. Then $C$ is closed as it is the intersection of a collection of closed sets. It remains to show $C$ contains no isolated points. Take any $x \in C$. Then $x \in E_n$, as above, for each $n \geq 1$. Hence for each $n \geq 1$ we see that $x$ lies in some closed interval $[a_{n,i}, b_{n,i}]$ of length $3^{-n}$ and for each $n \geq 1$, $a_{n,i}, b_{n,i} \in C$. So for each $n \geq 1$ set $x_n = a_{n,i}$ unless $x = a_{n,i}$, in which case we set $x_n = b_{n,i}$. Then we have, for all $n \geq 1$, that $x_n \in C$, $x_n \neq x$ and $|x - x_n| \leq 3^{-n}$. So $(x_n)_{n \geq 1}$ is a sequence of elements of $C$, distinct from $x$ and converging to $x$, so $x$ is not isolated. It follows that $C$ is perfect.

7. Let $S \subseteq \mathbb{R}^n$ be a perfect set. Since any closed subset of a complete metric space is complete we see that $S$ is complete. So it suffices to show that any complete metric space without isolated points in uncountable.

Clearly any finite metric space contains isolated points. Say $(X, d)$ is a countable complete metric space without isolated points. Write $X = \{x_n : n \geq 1\}$. Consider the sets $U_n = X \smallsetminus \{x_n\}$. Since $\{x_n\}$ is closed, $U_n$ is open. Since $x_n$ is not isolated, for all $\varepsilon > 0$, $B(x_n, \varepsilon) \cap U_n \neq \emptyset$, hence $U_n$ is dense in $X$ (any other point not equal to $x$ is already contained in $U_n$). By the Baire Category Theorem $\cap_{i=1}^{\infty} U_n$ is dense in $X$. But $\cap_{i=1}^{\infty} U_n = \emptyset$, a contradiction.