Building Geometric Structures:

Summary of Prof. Yau's lecture, Monday, April 2 [with additional references and remarks] (for people who missed the lecture)

A geometric structure on a manifold is a cover by coordinate systems [a "sub-atlas"] in which the transition functions from one coordinate system to another are not arbitrary C^∞ mappings but rather belong to some specific set of mappings from (open subsets of) Euclidean space to itself. [Precisely what is needed here is dealt with by the concept of "pseudo-group" of transformations]. Examples are

1. projective structures—transformations are (restrictions of) projective transformations of RP^n, namely linear fractional transformations

2. conformal structures: overlap maps are conformal

3. affine structures: overlap maps are affine transformations of Euclidean space

4. complex structures: overlap maps are holomorphic maps [considered as real C^∞ maps on an even dimensional Euclidean space]

These situations are different from each other and higher dimensions are often different from dimension 2. Important differences: Holomorphic maps do not have to come from a map of the whole space whereas, e.g., projective maps extend by definition. In dimension 2, there are many [infinite dimensional, not described by a finite set of parameters] maps [holomorphic functions] whereas in higher dimensions, conformal maps are a more rigid[extend to whole space except for possible definitely determined singularities, finite number of parameters, e.g., Liouville's Theorem that conformal maps of R^3 are generated by inversions and isometries and dilations/contractions].

Dimension 2: Riemannian metric always exists [manifolds are being assumed paracompact] and by historical theorems [Korn, Lichtenstein, early 1900s, etc.] there is a conformal structure attached to any given Riemannian metric, namely, there exist local coordinates [historically, “isothermal parameters”] in which the metric has the form (positive function) (coordinate Euclidean metric) ;and a cover by such coordinate systems gives a conformal structure, clearly. Assuming orientability, as is assumed from now on, this gives also a complex structure since conformal orientation-preserving maps are necessarily holomorphic in real dimension 2.

Almost complex structure: On a manifold with complex structure, we get a splitting at each point of the complexified tangent space [real tangent space tensored with the complex numbers] into two disjoint, spanning subspaces, with each the conjugate of the other. These are the holomorphic and antiholomorphic tangent spaces. This is the same as the span of the coordinate $\frac{\partial}{\partial z_j}$ operators, resp. $\frac{\partial}{\partial \overline{z}}$ operators, if $(z_1, \ldots z_n)$ is a holomorphic local coordinate system. . But the splitting also corresponds to eigenspaces of J (the real representation of multiplication by i on R^{2n}, J on tangent spaces is invariant under holomorphic maps), eigenvalues are +i and –i since J composed with J is multiplication by -1. Also we can recover J from the splitting [because of this eigenvalue observation]. So we can think about an “almost complex structure” , namely a suitable splitting or equivalently a J operator at each point with J composed with J being multiplication by -1. [“J squared= -1 ” for short] .

In two dimensions, every almost complex structure arises from a complex structure [e.g., one can J -average some Riemannian metric to make it J- invariant, then find the associated complex structure via the isothermal parameters idea , and the complex structure obtained has the same J as the original one.] From the general viewpoint from higher dimensions, this happens because $\left(\overline{\partial}\right)^2$ is always zero in dimension 2 on account of dimension reasons, while $\left(\overline{\partial}\right)^2 = 0$ is the condition for being able to attach a complex structure to an almost complex one.

In more detail, note that $\bar{\partial}$ is always defined as a operator, if just an almost complex structure is given: the almost complex structure gives forms (p,q) types;one gets $\bar{\partial}$ from there. The vanishing of the "Nijenhuis tensor"[see article on ~greene website, for course no 234] , is equivalent to $\left(\bar{\partial}\right)^2$ being zero. And by the fundamental Newlander Nirenberg Theorem, this implies the "integrability" of the complex structure. Here "integrability" means the existence of local coordinate systems in which the given J (or splitting of the complexified tangent space) is the same as that determined by the coordinate system considered as a map into complex Euclidean n-space.

In general, integrability of this sort is obtained in the real analytic case from the Cartan-Kaehler theory[a generalized version of the Cauchy-Kowaleska Theorem in which integrability conditions can be treated, but only in the real analytic case in general]. But in the case where real analyticity is not assumed, other methods are needed.[One approach to proving the Newlander-Nirenberg Theorem: work through solving $\bar{\partial}$ without using local coordinates, just using the J and integrability conditions to conclude that local "holomorphic relative to given J" functions exist in abundance].

Deformation especially in dimension 2:

Complex structures are described up to equivalence by 3g-3 complex parameters(known to Riemann). Here equivalence means that there is a structure preserving diffeomorphism from one to the other. In general, deformation of almost complex structure means tilting the holomorphic tangent space within the whole complexified tangent space. So one can think of this as a linear map (at each point) from the holomorphic tangent space into the conjugate (or antihlomorphic) tangent space. Tracing through identifications gives the infinitesimal deformation as represented by an element of the sheaf cohomology H^1(M, sheaf of germs of holomorphic vector fields) [Kodaira-Spencer theory, cf Morrow and Kodaira's book]. In the Riemann surface case, this becomes(via Serre Duality) the space of quadratic differentials , i.e., holomorphic sections of the square of the canonical(line) bundle (the bundle of (1,0) forms for a Riemann surface). [Note that {the square of }the canonical line bundle is a positive bundle precisely if the genus g is greater than one. It is negative and hence without nontrivial holomorphic sections if g=0 . This corresponds to the fact that deformations of CP^1 are trivial{CP^1 has a unique complex structure up to equivalence}. Nontrivial deformations of higher genus surfaces are abundant, e,g., the 3g -3 parameters when g>1, the one {complex} parameter when g=1. The 3g-3 number arises as follows: by the Riemann Roch Theorem, the dimension of the space of holomorphic sections of a line bundle = degree of bundle –g +1 + dim of space of holomorphic sections of (canonical bundle tensored with the dual of the given bundle). Here degree of bundle= total order of a meromorphic section = (first) Chern class interpreted as integer. Since the degree of the canonical bundle is 2g-2 and hence of the square of the canonical bundle is 4g-4, we get dimension of space of sections of the squared canonical bundle = 4g-4 –g+1 + dimension of space of sections of the dual of the canonical bundle. But the dual of the canonical bundle has degree 2-2g, which is <0 in case g>1, so the last dimension item is in fact 0. Thus the deformation space has dimension 3g-3 when g>1. The same argument, mutatis mutandis, explains the one(complex) parameter in the torus case, genus=1.]

Existence of almost complex structures:

For any n-manifold, the tangent bundle is obtained by pulling back the Grassmannian bundle of n-planes in a high dimensional Euclidean space via a map of the manifold into the Euclidean space. More explicitly, according to Whitney‘s embedding theorem, if M is a (smooth) manifold of dimension n then there is an embedding of M into a Euclidean space R^N dimension N>>n. Consider the Grassmannian Gr(n, N) of n planes in N-dimensional Euclidean space, with its tautological n-bundle, wherein the fibre over a point , a point being an n-plane, is that n-plane itself. Then the embedding induces a map of M into the Grassmannian Gr(n,N) by sending each point p in M to the image under the differential of the embedding at p applied to the tangent space of M at p. In other words, we send each point p to the tangent space of M at p , considered to be a subspace of R^N

when M is considered as being in R^N via the embedding. It is easy to see that the pullback of the tautological bundle of Gr(n,N) by this mapping of M into Gr(n,N) is in fact the tangent bundle of M!

This same construction can be extended to show that any n-plane bundle on M arises from pullback of the tautological bundle via a map into Gr(n,N) , N>>n. Moreover, this map is unique up to homotopy, for a given bundle. This means that the pullback of the cohomology classes of Gr(n,N) to M via this map depends only on the bundle itself. These are the characteristic classes of the bundle, by definition. Since this whole construction is independent of N when N is large, one usually lets N "go to infinity" and looks only at the "classifying space" $Gr(n,\infty)$. If the bundle has group G one writes B_G. (see Milnor, Characteristic Classes for details of all this). E.g., $Gr(n,\infty) = B_{Gl(n,\mathbb{R})}$ or , which is the same, $B_{O(n)}$.

Now an almost complex structure on a manifold of real dimension 2n amounts to a reduction of the structure group from GL(2n,R) to Gl(n, C) or, equivalently from the homotopy viewpoint, from SO(2n, R) to U(n) [orientability is automatic if an almost complex structure is to exist so we take it for granted]. For this to happen, the classifying map for the tangent bundle (a real 2n –plane bundle) has to factor through a map into the classifying space of U(n). Namely, the classifying space for U(n) has a natural "projection" map into the classifying space for SO(2n). This amounts to no more than observing that a complex subspace of complex dimension n of C^N can be considered to be a real subspace of real dimension 2n in R^{2N}. The almost complex structure then gives a way to "lift" the classifying map into $B_{SO(2n)}$. Namely, there is a map into $B_{U(n)}$ which when followed by the "projection" is the original "real" classifying map.

This can be translated into things about characteristic classes. The real characteristic classes, that is pullbacks from $B_{SO(2n)}$ of its cohomology classes, are generated by the Pontryagin classes p_1, p_2,… (integral classes) and the Stiefel-Whitney classes w_1, w_2,

…(mod 2 classes). The classes that come from $B_{U(n)}$ are the Chern classes (integral classes) c_1, c_2,…

For the existence of a lift as described, some relationships have to happen: for example, in complex dimension 2, it must be that $w_2 = c_1$ mod Z_2 while $p_1 = 2c_2 - c_1^{\ 2}$.
In algebraic topology (obstruction theory), it is known how to compute what has to happen for a lift to exist (up to homotopy) of the type that will correspond to an almost complex structure. In this case, this is all determined by characteristic classes. This has been worked out explicitly by Wu up to real dimension 6(and in outline 8) and in principle could be carried out for higher dimensions. But this is hard to do in general form (similar to the situation with homotopy groups of spheres).

And the problem of going from the almost complex structure (if there is one) to finding an integrable one is not solved in general. Only in dimension 2 is this automatic.

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