# Simpson's Rule Estimate

Preliminary: If $f$ is four times differentiable $f : \mathbb{R} \to \mathbb{R}$, then

$$\left| \int_{-h}^{+h} f(t)\,dt - \tfrac{1}{3}h\left(f(-h) + 4f(0) + f(h)\right)\right|$$

$$\le h^5 \sup_{|\lambda| \le h} |f^{(iv)}(\lambda)|$$

Proof: Write $S_h(f) = \frac{1}{3}h(f(-h) + 4f(0) + f(h))$

Then by direct calculation $\int_{-h}^{h} f(t)\,dt - S_h(f) = 0$ if $f$ is a polynomial of degree $\le 3$ (only $f(t) = t^2$ requires nonobvious calculation, really). For general $f$, write $f = f_3 + R$ where $f_3 =$ the Taylor Series of $f$ (at 0) up to and including order 3. Then

$$\int_{-h}^{+h} f(t)\,dt - S_h(f) = \int_{-h}^{+h} f_3(t)\,dt - S_h(f_3) + \int_{-h}^{+h} R(t)\,dt - S_h(R)$$

$$= \int_{-\infty}^{+\infty} R(t)\,dt - S_h(R) \quad \text{by the remark earlier.}$$

Now let $A = \sup_{\lambda \in [-h,h]} |f^{(iv)}(\lambda)| = \sup_{\lambda \in [-h,h]} |R^{(iv)}(\lambda)|$

By Taylor's Formula with Remainder,

$$\left| \int_{-h}^{+h} R(t)\,dt \right| \le \int_{-h}^{+h} \frac{A}{4!} t^4\,dt = \frac{2A}{5!} h^5 = \frac{A}{60} h^5$$

Also $$|S_h(R)| \le \left(\frac{A}{4!}h^4 + \frac{A}{4!}h^4 + 4\cdot 0\right)\frac{h}{3} = \frac{A}{36}h^5$$

Thus $$\left|\int_{-h}^{+h} R(t)\,dt - S_h(R)\right| \le \left(\frac{1}{36}+\frac{1}{60}\right)A\,h^5.$$

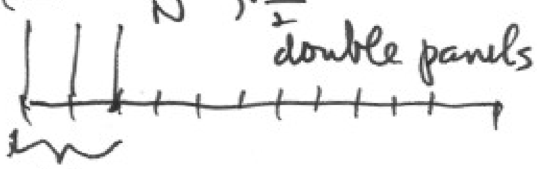
(NB This is not the best possible estimate. The best possible is actually $\left(\frac{1}{36}-\frac{1}{60}\right)A\,h^5 = \frac{A}{90}h^5$)

Now suppose $[a,b]$ is divided into $\frac{N}{2} = 2k/2$ "double panels". Then

$$\left|\int_a^b f(t)\,dt - \sum_{\text{double panels}} S_h(f)\right| \qquad h = \frac{b-a}{N}$$

$$\le \frac{N}{2}\,(\text{max diff in double panel})$$

$h = \frac{b-a}{N}$, $\frac{N}{2}$ double panels



$2\left(\frac{b-a}{N}\right)$

$$\le \frac{N}{2}\sup_{[a,b]}|f^{(iv)}| \cdot \left(\frac{1}{36}+\frac{1}{60}\right)\left(\frac{b-a}{N}\right)^5$$

$$\le C \sup_{t\in[a,b]} |f^{(iv)}(t)|\; N^{-4}$$

where $C$ depends only on $b-a$.

(Best possible $C$ is $\frac{1}{180}$. One we proved is $\frac{1}{2}\left(\frac{1}{36}+\frac{1}{60}\right)$ ).