The number

$$\frac{\mathrm{I}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)\mathrm{II}\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) - \mathrm{II}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)^2}{\left\langle\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right\rangle\left\langle\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right\rangle - \left\langle\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\rangle^2}$$

which is intrinsic and coordinate independent is called the Gauss curvature (varies point to point).

7. Show that if $S(u,v) = (u, v, f(u,v))$ and if $S(0,0) = (0,0,0)$ while $\frac{\partial f}{\partial u} = \frac{\partial f}{\partial v} = 0$ at $(0,0)$ then Gauss curvature at $(0,0) = \frac{\partial^2 f}{\partial u^2}\frac{\partial^2 f}{\partial v^2} - \left(\frac{\partial^2 f}{\partial u \partial v}\right)^2$

8. Define the "Gauss map" of $S \xrightarrow{\Gamma}$ unit sphere S^2 by $\Gamma(u,v) = \frac{S_u \times S_v}{\|S_u \times S_v\|}$. Show that the "determinant" of Γ = Gauss curvature.*
[Suggestion: It is enough to treat the case of problem 7!].

9. Show that (u,v) is a noncritical point of Γ if and only if Gauss curvature at $(u,v) \neq 0$ (using problem 8). Deduce that the image in S^2 of the set of points of S with Gauss curvature $= 0$ is measure 0 in S^2.

* Part of the problem is to make sense of "determinant" here.

*10. Suppose S is a compact surface (compact 2-dim. submanifold of $\mathbb{R}^3$). Of course this is probably (indeed surely) not covered by one coordinate patch. But everything still makes sense about Gauss map etc. (you may assume S is orientable: this is actually automatically true). Use Morse theory for the function $p \to \langle S(p), N_0 \rangle = \langle p, N_0 \rangle$ (we think of $p \in S$ and $S \subset \mathbb{R}^3$) where N_0 is $\in S^2$ & not in the image of Γ applied to points of S with Gauss curvature zero to try to prove that

$$\deg \Gamma = \frac{1}{2} \chi(M)$$

($\chi(M)$ = Euler characteristic of M).

Then use pull-back by Γ of volume form of S^2 to get

$$\int_S K \, d(\text{area}) = \frac{1}{2}(4\pi \, \chi(M)) = 2\pi \, \chi(M).$$

Note: This is an open-ended "extra credit" problem.