Simple Connectivity and Solution of $df = \omega$ given $d\omega = 0$.

Definition: An open set U in a manifold M is a coordinate ball if there is a coordinate system $\varphi: U \to \mathbb{R}^n$, $n = \dim M$, such that $\varphi(U)$ is an open ball in $\mathbb{R}^n$.

Lemma: If U is a coordinate ball and ω is a 1-form on M with $d\omega \equiv 0$, then $\exists\, f: U \to \mathbb{R}$ such that $df \equiv \omega$ on U.

Proof: It suffices, since $d = 0$ is a diffeomorphism-invariant condition to show that if ω is a 1-form on an open ball around $\vec{0}$ in $\mathbb{R}^n$ with $d\omega \equiv 0$, then $\exists f$ on the same open ball with $df \equiv \omega$. The construction of f is by the usual "over and up" integration process

$$f(x_1, \ldots x_n) = \int_0^{x_1} p_1(t_1, 0 \ldots 0)\, dt_1 + \int_0^{x_2} p_2(x_1, t_2, 0 \ldots 0)\, dt_2$$

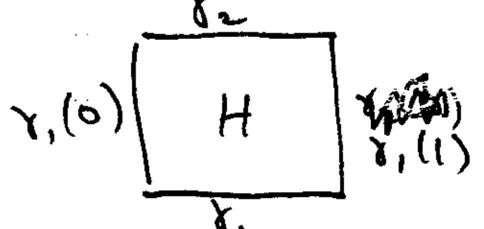
$$+ \int_0^{x_3} p_3(x_1, x_2, t_3, 0 \ldots 0)\, dt_3 + \ldots + \int_0^{x_n} p(x_1, x_2 \ldots x_{n-1}, t_n)\, dt_n$$

where $\omega = p_1 dx_1 + \ldots + p_n dx_n$.
That $df = \omega$ is established by differentiation under the $\int$ signs using repeatedly $\frac{\partial p_i}{\partial x_j} = \frac{\partial p_j}{\partial x_i}$. (Calculus exercise). $\square$

The Lemma gives a sufficient condition for the solution of $df = \omega$ given $\omega = 0$. But a much weaker condition suffices to guarantee the existence of a solution:

Theorem: If M is simply connected and if ω is a 1-form on M with $d\omega \equiv 0$, then $\exists$ $f: M \to \mathbb{R}$ such that $df \equiv \omega$.

[Recall that M is <u>simply connected</u> if and only if for each pair of continuous curves $\gamma_1 : [0,1] \to M$, $\gamma_2 : [0,1] \to M$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$, there is a continuous function $H(t,s)$, $H: [0,1] \times [0,1] \to M$ such that $H(0,s) = \gamma_1(0)$, $\forall s \in [0,1]$, $H(1,s) = \gamma_1(1)$, $\forall s \in [0,1]$, and $H(t,0) = \gamma_1(t)$, $\forall t \in [0,1]$ and $H(t,1) = \gamma_2(t)$, $\forall t \in [0,1]$.
Pictorially we represent these conditions by



This is a purely topological condition (no differentiability in sight). It is equivalent to all closed curves are free-homotopic to a constant and also to $\pi_1 = 0$ at any base point: cf. Gamelin & Greene, Intro to Topology, Chapter 3, for example].

For the proof of the Theorem, we introduce the idea of integrating a closed differential 1-form along a continuous path (the usual line integral definition requires a piecewise C^1 path). For this definition, let $\gamma : [a, b] \to M$ be a continuous function. Now for each $t \in [a, b]$, $\gamma(t)$ lies in some coordinate ball; choose one and denote it by U_t. Then $\gamma^{-1}(U_t)$ is open in $[a, b]$ by continuity of γ. Also $[a, b] \subset \bigcup_{t \in [a,b]} U_t$

since $t \in U_t$, $\forall t \in [a, b]$.

Now recall the Lebesgue Covering Lemma: If X is a compact metric space and U_λ, $\lambda \in \Lambda$ is an open cover of X, then $\exists$ $\alpha > 0$ such that for all $x \in X$, $B(x, \alpha)$ is contained in some one of the sets U_λ, i.e., $\exists$ λ_x such that $B(x, \alpha) \subset U_{\lambda_x}$. The proof of this is a straightforward argument by contradiction: Suppose successively that $\alpha = 1, \frac{1}{2}, \frac{1}{3} \ldots$ each fail so $\exists$ for each $n = 1, 2, 3 \ldots$, a point x_n such that $B(x_n, 1/n)$ fails to be contained in a single U_λ. Using compactness of X, there is a subsequence x_{n_j} converging to a limit x_0. Now $x_0 \in U_{\lambda_0}$ since the U's cover X. But since U_{λ_0} is open,

$B(x_{n_j}, 1/n_j) \subset U_{j_0}$ (by the triangle inequality) for all j sufficiently large, contradicting the choice of the x_n's. $\square$

With these preliminaries in sight, we now define with $d\omega \equiv 0$ the integral

$$\oint_\gamma \omega$$

as follows: Choose a finite partition $a = a_0 < a_1 < \ldots < a_N = b$ of $[a,b]$ such that $\gamma([a_i, a_{i+1}]) \subset$ some coordinate ball U_i for each $i = 0, 1, \ldots, N-1$. (This is possible: one need only choose $\max_{i=0,\ldots N-1} |a_{i+1} - a_i| <$ an α that satisfies the Lebesgue covering lemma condition for the open cover $\bigcup_{t \in [a,b]} U_t$ of $[a,b]$.

Choose $f_i : U_i \to \mathbb{R}$, $i = 0, \ldots, N-1$ such that $df_i = \omega$ on U_i. Finally set

$$\oint_\gamma \omega = \sum_{i=0}^{N-1} f_i(\gamma(a_{i+1}) - f_i(\gamma(a_i))].$$

It is a tedious but routine exercise to show that this definition is independent of all the choices made, in particular of the choice of the partition of $[a,b]$ satisfying the condition indicated.

The basic result relating this concept and simple connectivity is this:

Lemma: If M is simply connected and if ω is a 1-form with $d\omega \equiv 0$ on M (i.e., M is closed) then for any two continuous curve $\gamma_1 : [0,1] \to M$ and $\gamma_2 : [0,1] \to M$ with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(1) = \gamma_2(1)$,

$$\oint_{\gamma_1} \omega = \oint_{\gamma_2} \omega$$

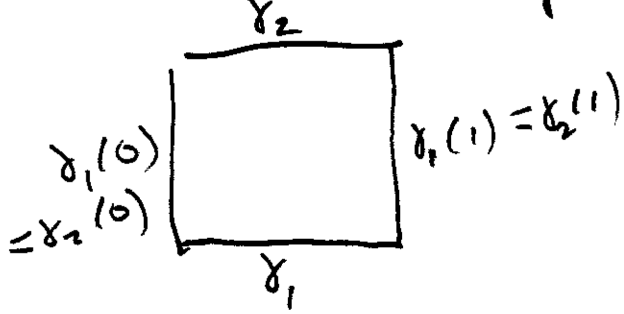
Assuming this Lemma for a moment, the following result is obtained almost immediately:

Theorem: If M is simply connected, then every closed 1-form on M is exact, i.e. if $d\omega \equiv 0$, ω a 1-form, then $\exists\, f : M \to \mathbb{R} \ni df = \omega$.

Proof of the Theorem from the Lemma: Pick a point $p_0 \in M$ and set $f(p_0) = 0$. For each $p \in M$, $p \neq p_0$, define $f(p) = \oint_\gamma \omega$ for some $\gamma_p : [0,1] \to M$ with $\gamma_p(0) = p_0$, $\gamma_p(1) = p$. According to the Lemma, f is defined independently of the choices of the curves γ_p for different points p. Moreover the Lemma also implies that, in a coordinate-ball neighborhood U_{p_1} of any chosen $p_1 \in M$, $p_1 = \vec{0}$, $f(q) = f(p_1) + \oint_\sigma \omega$ where σ = the curve in U_{p_1} which in the coordinate system is the "over-and-up" piecewise C^1 path in the ball used to find a "local f" for ω on U_{p_1}. In particular $df = \omega$ on U_{p_1}. □

p_1 can be taken to be the center of the coordinate ball, which can be taken to be $\vec{0} = (0, \ldots, 0)$ in $\mathbb{R}^n$

We turn now to the proof of the Lemma:
By simple connectivity, there is a homotopy H with fixed endpoints as before: $H : [0,1] \times [0,1] \to M$.

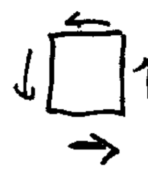


The Lebesgue Covering Lemma implies as before that there is a subdivision of $[0,1]$ $\quad a_0 = 1 < a_1 < \dots < a_N = 1$

such that, for all $i, j \in \{0, \dots, N-1\}$,
$H([a_i, a_{i+1}] \times [a_j, a_{j+1}]) \subset$ some coordinate ball.
In particular, for each pair i, j

$$\oint_{H|[a_i, a_{i+1}] \times a_j} \omega + \oint_{H| a_{i+1} \times [a_j, a_{j+1}]}$$

$$- \int_{H|[a_i, a_{i+1}] \times a_{j+1}} \omega - \int_{H|a_i \times [a_j, a_{j+1}]} \omega = 0$$

Schematically,  the integral of ω around the boundary of the box $[a_i, a_{i+1}] \times [a_j, a_{j+1}]$ $= 0$. But adding all these 4-fold sums together yields

$$\oint_{\gamma_1} \omega - \oint_{\gamma_2} \omega = 0$$

(since the inner edges cancel and the vertical sides ~~as~~ $(a_0, \cdot)$ and $(a_N, \cdot)$ have H constant along them. $\square$