A Quick Tour of Basic Riemannian Geometry

Riemann metric on manifold M is (def.) an assignment to each $p \in M$, a symmetric, positive definite bilinear form $g(\cdot,\cdot\cdot)$ or $<\cdot,\cdot\cdot>$ on T_pM. The metric is C^∞ (def.) if all $g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ is C^∞, $(x_1, .. x_n)$ local coordinates. Exercise: Enough to check coordinate cover: $g_{ij}\ (= g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}))$ $C^\infty \Leftrightarrow \hat{g}_{ij} (= g(\frac{\partial}{\partial \hat{x}_i}, \frac{\partial}{\partial \hat{x}_j}))$ is C^∞, $(\hat{x}_1, .. \hat{x}_n)$ some other local coordinates C^∞ related on same domain.

Pair (M, g) is called a Riemannian manifold.

Fact: Every manifold has a Riemannian metric

Reason: $i : M \to \mathbb{R}^N$ immersion $\Rightarrow <v,w> \overset{def}{=} <i_* v, i_* w>_{\mathbb{R}^N}$ determines Riemannian metric on M.

(Nash Isometric Embedding Theorem: Every Riemannian metric arises this way, for some suitable choice of i).

Alternative proof of $\exists$ of Riemannian metric: Choose $\varphi_\lambda : U_\lambda \to \mathbb{R}^n$ coordinate curve, choose partition of unity ρ_λ subordinate to $\{U_\lambda\}$ and set $g = \sum \rho_\lambda g_\lambda$ where $g_\lambda(v,w) = <(\varphi_\lambda)_* v, (\varphi_\lambda)_* w>_{\mathbb{R}^n}$ if $v, w \in T_pM$, $p \in U_\lambda$.

(M, g) Riemannian manifold, $\gamma : [a, b] \to M$ piecewise C^∞, $\ell(\gamma) \overset{def}{=} \int_a^b <\gamma'(t), \gamma'(t)>^{\frac{1}{2}}\, dt$, where $<\gamma', \gamma'>^{\frac{1}{2}} =$ length of γ' (same def for vectors in general $\|v\| = <v, v>^{\frac{1}{2}}$).

Def: $$\mathrm{dis}_M(p, q) = \inf_{\substack{\gamma(0)=p \\ \gamma(1)=q}} \ell(\gamma) \qquad \gamma : [0, 1] \to M$$

dis_M is a metric space distance function on M. All properties obvious except $\mathrm{dis}_M(p,q)=0 \Rightarrow p=q$. This follows since if $\varphi: U \to \mathbb{R}^n$ is a coordinate system around p, $g|U$ and $\langle \varphi_* v, \varphi_* w\rangle_{\mathbb{R}^n}$ are uniformly comparable on some (compact closure $\subset U$) neighborhood of p. Exercise for rest, also metric space topology = manifold topology by same uniform comparability

$\mathrm{dis}(p,q)$ need not be realized by some particular γ, e.g. $M = \mathbb{R}^2 - \{(0,0)\}$, euclidean metric, $p=(-1,0)$, $q=(1,0)$

A curve $\gamma: [a,b] \to M$ is <u>minimal</u> if $\mathrm{dis}(\gamma(a),\gamma(b)) = \ell(\gamma)$
Then γ minimal $\Rightarrow$ $\gamma|[c,d]$ is minimal, any $[c,d] \subset [a,b]$.
Calculus of variations $\Rightarrow$ γ minimal satisfies a differential equation (Euler-Lagrange eq.). Specific equation is most easily formulated by generalizing idea that (arclength parameter) straight line in $\mathbb{R}^n$ has "acceleration 0". So look for natural idea of acceleration and hope acceleration 0 is necessary condition for minimal (up to parameterization at least).

Acceleration / vector field differentiation in $\mathbb{R}^n$ is component by component: If $Y = \sum b_j \frac{\partial}{\partial x_j}$, b_j functions, v a vector then

$$D_v Y = \sum (v b_j) \frac{\partial}{\partial x_j}$$

and acceleration of $(x_1(t), \ldots, x_n(t))$

$$= \sum_j \frac{d^2 x_j(t)}{dt^2} \frac{\partial}{\partial x_j}$$

These are not coordinate invariant (e.g. $\frac{\partial}{\partial\theta}$ in (r,θ) coordinates does not have $D_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = 0$ if computed in (x,y) coordinates, even though $\partial/\partial\theta$ is constant coefficient in (r,θ) terms).

Approach to finding a coordinate invariant D operator given a Riemannian metric:

Note that on $\mathbb{R}^n$: (1) $D_X Y - D_Y X = [X,Y]$, X, Y vector flds

(2) $X\langle Y, Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$ (easy check Leibnitz Rule)

Look at $X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle$

$$= \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle + \langle D_Y X, Z\rangle + \langle X, D_Y Z\rangle - \langle D_Z X, Y\rangle - \langle X, D_Z Y\rangle$$

$$= \langle D_X Y + D_Y X, Z\rangle + \langle Y, D_X Z - D_Z X\rangle + \langle X, D_Y Z - D_Z Y\rangle$$

$$= 2\langle D_X Y, Z\rangle - \langle [X,Y], Z\rangle + \langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle$$

and use this to <u>define</u> $D_X Y$ namely

$D_X Y$ is vector such that

$$\langle D_X Y, Z\rangle = \frac{1}{2}\big(X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle + \langle [X,Y], Z\rangle - \langle Y, [X,Z]\rangle - \langle X, [Y,Z]\rangle\big)$$

where RHS is independent of D, depends only on metric and Lie bracket.

Long exercise: RHS is func. linear in Y and Z, Leibnitzian in Y ($\mathrm{RHS}(fY) = (Xf)\langle Y, Z\rangle + f\,\mathrm{RHS}(Y)$) with X, Z fixed

So RHS makes definition of $D_v Y$ with expected properties.
Acceleration definition comes from this: $\gamma(t) = (x_1(t) \ldots x_n(t))$ ~~that~~ then

$$\gamma''(t) = \sum_j \frac{d^2 x_j(t)}{dt^2}\frac{\partial}{\partial x_j} + \sum \frac{dx_j}{dt}\frac{dx_\ell}{dt} D_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_\ell}$$

(coordinate invariant).
Fact: γ minimal $\Rightarrow$ $\gamma'' \equiv 0$ (after reparam. if necessary)
(Converse true only locally)
$\gamma'' \equiv 0 \Leftrightarrow$ (definition) γ is a <u>geodesic</u>

Fact: (from diff. eqs) Given $\gamma(0), \gamma'(0)$ hoped-for values, $\exists$ γ (on some interval around 0, $(-\varepsilon, \varepsilon)$) with γ geodesic and $\gamma(0), \gamma'(0)$ as hoped for.
Uniqueness, too (on common interval of definition for two such)

Hopf Rinow Theorem: (M, dis) Cauchy complete $\Rightarrow$
Given $p, q \in M$, $\exists$ $\gamma_{p,q}$ with endpoints p, q, γ minimal geodesic, i.e. $\ell(\gamma) = \text{dis}(p,q)$.
Geodesics look locally, from a point, like st. lines from a pt. in $\mathbb{R}^n$
How geodesics "spread" in detail is controlled by "curvature": long story for later.
(Curvature involves second derivatives of g_{ij}'s).