Homework 3: A Mini-course in one complex variable using differential forms.

1. A C^∞ function $f: U \to \mathbb{C}\ (= \mathbb{R}^2)$, $U \subset \mathbb{C}\ (= \mathbb{R}^2)$ is holomorphic if, writing $f = u + iv$, $u, v : U \to \mathbb{R}$,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{(Cauchy Riemann equations)}.$$

Given a C^∞ curve $\gamma : [a, b] \to U$ and $f : U \to \mathbb{C}$ a (not necessarily holomorphic) function, the complex line integral is defined by

$$\oint_\gamma f\, dz \overset{\text{def.}}{=} \left(\int_a^b u \frac{dx(t)}{dt} - v \frac{dy(t)}{dt}\, dt\right) + i\left(\int_a^b v \frac{dx(t)}{dt} + u \frac{dy(t)}{dt}\, dt\right)$$

where $\gamma(t) = (x(t), y(t))$ and $f = u + iv$.

[Definition is motivated by $(u+iv)(dx + i\,dy)$
$= u\,dx - v\,dy + i(v\,dx + u\,dy)$].

Prove (a) If $f = u + iv$ is holomorphic, $\frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$, $\frac{\partial^2 v}{\partial x^2} = -\frac{\partial^2 v}{\partial y^2}$

(b) If f is holomorphic on U, then $\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$ is holomorphic on U.

Prove: (b1) If f, g are holomorphic then $f+g$, $f \cdot g$ and f/g (where $g \neq 0$ in last case) are holomorphic

(c) If f is holomorphic, then

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right) dz$$

(d) If f is holomorphic then $f(z)\,dz$
$= (u\,dx - v\,dy) + i(u\,dy + v\,dx)$
is closed (in the sense that its real and imaginary parts are closed).

(e) $\displaystyle\oint_{\text{circle around } 0} \frac{1}{z}\, dz = 2\pi i$

(f) If f is holomorphic on U and $\{z : |z| \le R\} \subset U$, then $\displaystyle\oint_{\gamma_R} \frac{f(z)}{z} dz = \oint_{\gamma_\varepsilon} \frac{f(z)}{z} dz$ where $\varepsilon < R$,
(continues)

and γ_ε = counterclockwise circle around $\vec{0}$ of radius ε.

(g) If f is holomorphic as in problem (f), prove that $f(z) = f(0) + z f'(0) + o(|z|)$ where $f'(0) = \left(\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}\right)\Big|_{(0,0)}$.

(h) Prove that $\displaystyle\lim_{\varepsilon \to 0^+} \oint_{\gamma_\varepsilon} \frac{f(z)}{z}\,dz = 2\pi i\, f(0)$

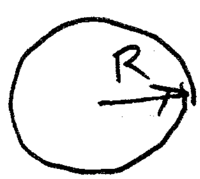
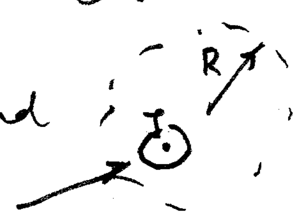
(i) Deduce that $\displaystyle f(0) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(z)}{z}\,dz$

(j) Generalize (i) to show that

$$f(z) = \frac{1}{2\pi i} \oint_{\gamma_R} \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

for every z with $|z| < R$.

(Suggestion: Recall that $f(\zeta)/(\zeta - z)$ is a holomorphic function of ζ on $\{\zeta : \zeta \in U, \zeta \neq z\}$ and use that $\int_{\text{closed curve}}$ closed differential does not change under homotopy of the closed curve so

$\oint$ around  $= \oint$ around 

circle rad ε around ζ).

(k) Prove: If U is simply connected and $f : U \to \mathbb{C}$ is holomorphic, then $\exists\ F : U \to \mathbb{C}$ holomorphic with $F' \equiv f$ on U.

2. Continuing problem 1, prove that if f is holomorphic on $\mathcal{U}$ with $\{z : |z| \le R\} \subset \mathcal{U}$, then $\exists$ $a_0, a_1, a_2 \ldots$ such that

$$\sum_{j=0}^{+\infty} a_j z^j$$ converges to $f(z)$ for all z with $|z|<R$.

(Suggestion $\dfrac{f(\zeta)}{\zeta - z} = \dfrac{f(\zeta)}{\zeta} \dfrac{1}{(1-\frac{z}{\zeta})} = \dfrac{f(\zeta)}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \ldots\right)$

if $|\zeta| = R$, $|z|<R$. Integrate term by term over γ_R).

3. Suppose $u : \mathcal{U} \to \mathbb{R}$ is a C^∞ function, $\mathcal{U}$ simply connected with $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ (u is <u>harmonic</u> by definition).

(a) Show that $\dfrac{\partial u}{\partial x} - i \dfrac{\partial u}{\partial y}$ is holomorphic.

(b) Deduce that there is a holomorphic function $F : \mathcal{U} \to \mathbb{C}$ with $F = U + iV$ such that $F' = \dfrac{\partial u}{\partial x} - i \dfrac{\partial u}{\partial y}$.

not $\mathcal{U}$! This is a function

(c) Conclude that u is the real part of some holomorphic function on U.
(Suggest: U has the same partial derivative as u).

(d) Show by example that this (part c) may not work if $\mathcal{U}$ is not simply connected.