The deRham Isomorphism for p-forms, $p>1$

To extend our ideas for the $p=1$ case to $p>1$, we need first to extend the idea of Cech cohomology. For this, suppose $\{U_i : i \in \Lambda\}$ is a covering with U_i's connected and all nonempty intersections of any order ($\leq n+1$, n = dimension M) connected. A Cech cochain of order p, or p-Cech-cochain, is an assignment, to each ordered $(p+1)$-tuple $i_0, i_1, \ldots, i_p$ with $U_{i_0} \cap \ldots \cap U_{i_p} \neq \phi$, a number $\alpha_{i_0 \ldots i_p}$ in such a way that $\alpha_{i \ldots i}$ is antisymmetric in its indices:

$$\alpha_{i_0 \ldots i_p} = -\alpha_{i_0 \ldots i_j \ldots i_l \ldots i_p} \quad \text{for } l<j \text{ (two indices}$$

i_l and i_j interchanged). We have defined $\mathbb{R}$-valued p-cochains, but a similar definition could be made where $\alpha_{i_0 \ldots i_p}$ was a form on $U_{i_0} \cap \ldots \cap U_{i_p}$, for example. All that is needed for the cohomology definition that follows is that whatever α is, one can add and subtract them and restrict them to smaller sets (i.e. if α is defined on $p+1$fold intersections we need to be able to restrict the α to a $p+2$ fold one). This will be clear as we continue.

A p-cochain is a cocycle if $\displaystyle\sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i_j} \ldots i_{p+1}} = 0$ (i_j deleted)

for all $(p+2)$ tuples $i_0 \ldots i_{p+1}$ such that $U_{i_0} \cap \ldots \cap U_{i_{p+1}} = 0$. The operator that takes $\{\alpha_{p+1 \text{ tuples}}\}$ (p cochain) to the $(p+1)$ cochain assigning

$$\sum_{j=0}^{p+1} (-1)^j \alpha_{i_0 \ldots \hat{i_j} \ldots i_{p+1}} \quad (i_j \text{ deleted}) \quad \text{to } i_0 \ldots i_{p+1} \text{ (when } U_{i_0} \cap \ldots \cap U_{i_{p+1}} \neq \phi)$$

is called the coboundary operator, denoted δ_p. So a p-cocycle is (by definition) a p-cochain with coboundary 0. (Compare to our previous $p=1$ discussion: to be a cocycle, a 1-cochain need to satisfy $\alpha_{ij} + \alpha_{jk} + \alpha_{ki} = 0$ ($\{\alpha_{ij}\}$ being the cochain).

This arises as $(\delta\alpha)_{ijk} = \alpha_{\hat{i}jk} - \alpha_{i\hat{j}k} + \alpha_{ij\hat{k}}$

0th element deleted; 1th element deleted; 2nd element deleted

$= +\alpha_{jk} - \alpha_{ik} + \alpha_{ij}$

$= +(\alpha_{ij} + \alpha_{jk} + \alpha_{ki})$, the vanishing of which is the same as our original "cocycle condition).

It is immediate to check that "$\delta^2 = 0$": if $\{\alpha\}$ is a p-cochain $\delta_{p+1}(\delta_p\{\alpha\}) = 0$. ($p+1$) tuple subscript

Definition: The Cech p-cohomology for the covering $\{U_i : i \in \Lambda\} = \dfrac{\text{kernel of } \delta_p \text{ (= coboundary on } p \text{ cochains)}}{\text{image of } \delta_{p-1} \text{ (= coboundary on } p-1 \text{ cochains)}}$

or, as one says for short, $\dfrac{p\text{-cocycles}}{p\text{-coboundaries}}$

This makes sense for cochains with $\mathbb{R}$-values, or values in differential forms of a fixed degree, or closed differential forms of a fixed degree. Note that δ would not chain degrees of forms: it is strictly combinatorial.

We can now state the basic deRham Isomorphism result:

deRham Isomorphism Theorem: Suppose $\{U_i : i \in \Lambda\}$ is a locally finite open cover of a differentiable manifold of M such that all nonempty U intersections involving $n+1$ or fewer U_i's are connected and have $H^k_{deR}(\cap, \mathbb{R}) = 0$ for $k \geq 1$. Then for each $k \geq 1$ ($k \leq n$)

$H^k_{deR}(M, \mathbb{R}) \cong$ Cech k-cohomology for the cover (with $\mathbb{R}$-values for the cochains).

The proof is by the same general method as for the $h=1$ case: Start with a k-form that is closed, and try to generate items associated to intersections. Specifically:

Suppose ω is a closed k-form. Then for each i, $\exists\ \theta_i$ such that $\omega|_{U_i} = d\theta_i$: the θ_i's are $k-1$ forms on U_i. Thus for i, j such that $U_i \cap U_j \neq \phi$, the $(k-1)$ form on $U_i \cap U_j$ defined by $\theta_i - \theta_j$ is closed since $d(\theta_i - \theta_j) = \omega - \omega = 0$. So associated to ω and the choice of θ_i's we get a collection, call it $\{\theta_{ij}\}$ of closed $(k-1)$-forms on the $U_i \cap U_j \neq \phi$ by setting $\theta_{ij} = \theta_i - \theta_j$. Think of this collection $\{\theta_{ij}\}$, which is antisymmetric in its indices (obviously) as a Cech 1-cochain with values in closed $(k-1)$ forms. It is easy to check that the Cech coboundary of $\{\theta_{ij}\} = 0$ because $\theta_{ij} = \theta_i - \theta_j$.

But $\{\theta_{ij}\}$, while a cocycle, may not be a coboundary of a Cech 0-cochain with values in closed (!) $(k-1)$-forms: the θ_i are not necessary closed and maybe one cannot find closed forms $\{\psi_i\}$ that satisfy $\theta_{ij} = \psi_i - \psi_j$. Working as before, it is not hard to check that this construction gives a map deRham k-classes $\longrightarrow$

Cech 1-cohomology classes with values in $(k-1)$ forms that are closed.

(It is vital here that the Cech classed

have values in <u>closed</u> $(k-1)$ forms.

The same partition-of-unity idea we used before gives an inverse map for this deRham → Cech map (Given θ_{ij}, a 1-cocycle with values in closed forms, we look at, on each U_i, the form $d(\sum_\ell \rho_\ell \theta_{\ell i})$, which gives a global k-form on M since $\sum \rho_\ell \theta_{\ell i} - \sum \rho_\ell \theta_{\ell j}) = \sum \rho_\ell(\theta_{\ell i} - \theta_{\ell j}) = \sum \rho_\ell \theta_{ij} = \theta_{ij}$ and θ_{ij} is closed on $U_i \cap U_j$).

This can be check to induce a well-defined map on Cech 1-cohomology classed with values in closed $(k-1)$ forms back to deRham k-classes on M

So deRham k-cohomology of M $\cong$ Cech 1-cohomology with values in closed $(k-1)$ forms.

This process can be continued in what one hopes is a clear pattern to show that

Cech 1-cohomology with values in closed $(k-1)$ forms $\cong$ Cech 2-cohomology with values in closed $(k-2)$ forms.

Eventually, the process terminates with a final isomorphism to Cech k-cohomology with values in closed 0-forms = Cech k-cohomology with number values ($\mathbb{R}$-values).

The question then arises, what kind of topological condition on the cover suffices to guarantee deRham k-vanishing for all intersections, $1 \le k \le n$?
If we had some realizable topological condition, then we could prove as before that two homeomorphic differentiable manifolds had isomorphic deRham cohomologies.
The right condition is: each nonempty intersection needs to be (continuously) contractible. (A topological space X is continuously contractible if $\exists$ a continuous map $H: X \times [0,1] \to X$ such that $H(\cdot, 0) =$ a constant point in X and $H(\cdot, 1) =$ the identity map of X). It can be shown by approximation techniques that continuous contractible $\Rightarrow$ smoothly contractible for open subsets of differentiable manifolds.
And we already know (Poincare Lemma) that smoothly contractible $\Rightarrow$ k-deRham $= 0$, $k \ge 1$.
So our program is complete (modulo proving continuous contractible $\Rightarrow$ smoothly contractible) of showing that deRham cohomology is topological — provided that we can show that every differentiable manifold has a "continuously contractible" (all intersections) covering! (later)