Computing deRham Cohomology [Notes: Sometimes I leave out $\mathbb{R}$ and write $H^k(M,\mathbb{R})$ as just $H^k(M)$ [all cohomology is deRham here]]

Two basic tools:

(1) Mayer-Vietoris long exact sequence

$$\to H^k(U\cup V) \to H^k(U)\oplus H^k(V) \to H^k(U\cap V) \to H^{k+1}(U\cup V) \to$$

(2) Homotopy invariance:

<u>Theorem</u> If $F : M_1 \times [0,1] \to M_2$ (notation $F_t(p)$, $p \in M_1$, $t \in [0,1]$) is C^∞, then $F_1^* = F_0^*$

where

(i) $F_1 : M_1 \to M_2$ is defined by $F_1(p) =$ F evaluated on $(p,1)$

and similarly for F_0

(ii) F_1^* and F_0^* are the induced maps from $H^k_{dR}(M_2)$ to $H^*_{dR}(M_1)$.

Proof of Theorem will be given later.

Backstory on induced map of deRham cohomology (cf 2009 notes for details)

ω closed k form on M_2, $G : M_1 \to M_2$

Set $G^*[\omega] = [G^*\omega]$ where $G^*\omega =$ pullback of ω to M_1 by G.

This is defined because (a) $d\, G^*\omega = G^*(d\omega) = G^*(0) = 0$

and well defined since

if $\hat{\omega} = \omega + d_{M_2}\theta$ then $G^*\hat{\omega} = G^*\omega + G^*(d_{M_2}\theta)$

$= G^*\omega + d_{M_1}(G^*\theta)$ so $[G^*\omega] = [G^*\hat{\omega}]$

Here we are using the basic fact that $G^* d_{M_2} = d_{M_1} G^*$.
[For detailed proof see 2009 notes on line

Basic idea: works on functions: $G^*(df) = d(G^*f)$
with $G^*f = f(G(\cdot))$. Also G^* commutes with $\wedge$
obviously. Leibnitzian property of d_{M_1} and d_{M_2} then
implies that d and G^* commute for all forms]

Important examples of homotopy idea:

$M = N \times (-1,1)$ $\quad F_t(\cdot) : M \to M$
defined by $F_t((p,s)) = (p, ts)$ $\quad p \in N$,
$t \in [0,1]$, $\quad s \in (-1,1)$
So $F_1 =$ identity map of M to M
$F_0 =$ map taking (p,s) to $(p,0)$.

Note that: If N has dimension $n-1$, so $M = N \times (-1,1)$
dimension n, then F_0^* on n forms is
the 0-map (since F_0^* has image of dimension
$= n-1$). So

$$H^n_{deR}(M, \mathbb{R}) = 0$$

because $F_1^* =$ identity on $H^n_{deR}(M,\mathbb{R})$ while $F_0^* = 0$
on $H^n_{deR}(M,\mathbb{R})$.

In particular, we get that

(2') $H^2(S^1 \times (-1,1), \mathbb{R}) = 0$.

Now we turn to how to compute $H^*(S^2, \mathbb{R})$ using the long exact sequence (1) and the application (2') of homotopy invariance (item (2)) given just above here. The sequence a la (1) we want is constructed using

$$U = \{(x,y,z) \in S^2 : z > -\tfrac{1}{4}\}$$
$$V = \{(x,y,z) \in S^2 : z < \tfrac{1}{4}\}$$ so that

$$U \cap V = \{(x,y,z) \in S^2 : z \in (-\tfrac{1}{4}, \tfrac{1}{4})\} \cong S^1 \times (-1,1)$$

diffeomorphic

(send $(x,y,z) \to \left(\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, 4z\right)$)

Then $H^0(U) = H^0(V) = H^0(U \cap V) = \mathbb{R}$ and the sequence becomes, since $H^0(U \cup V) = \mathbb{R}$ also:

$$H^0(U \cup V) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to H^1(U \cup V)$$
$$\mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \to \mathbb{R}$$
$$a \to (a, a)$$
$$(a, b) \to a - b$$
$$c \to c$$

So

$$\mathbb{R} \to \mathbb{R} \oplus \mathbb{R} \xrightarrow{\text{surjective}} \mathbb{R} \to$$

injective   image here is 0

is the first part. (This is how it always is if everything is connected. In particular,

$$H^1(U\cup V) \xrightarrow{\substack{\text{This is}\\ \text{injective}}} H^1(U)\oplus H^1(V) \quad \text{so} \quad H^1(S^2)=0.$$
$$H^1(U\cup V) = H^1(S^2), \quad H^1(U)=0, \quad H^1(V)=0$$

Also

$$H^1(U)\oplus H^1(V) \longrightarrow H^1(U\cap V) \to H^2(U\cup V) \to H^2(U)\oplus H^2(V)$$
$$H^1(U)=0,\ H^1(V)=0, \quad H^2(U\cup V)=H^2(S^2), \quad H^2(U)=0,\ H^2(V)=0 \text{ by homotopy principle (exercise)}$$

So $\quad H^1(U\wedge V) \cong H^2(S^2)$

Now what is $H^1(U\cap V)$? $(= H^1(S^1\times(-1,1)))$.
We can find this in various ways. You did this concretely in your Hwk VI. We can also do it by another exact sequence argument:
Let $\quad U_1 = (S^1\setminus\{(1,0)\})\times(-1,1)$, $V_1=(S^1\setminus\{(-1,0)\})\times(-1,1)$

$U_1 =$  $\times(-1,1)$, $V_1 =$  $\times(-1,1)$ So $U_1\cap V_1 =$ disjoint union of two open rectangles

Then

$$H^0(U_1\cup V') \xrightarrow{\text{injective}} H^0(U_1)\oplus H^0(V_1) \xrightarrow{\text{image has dim 1}} H^0(U_1\cap V_1) \longrightarrow H^1(U_1\cup V_1)$$
$$\mathbb{R} \qquad \mathbb{R}\oplus\mathbb{R} \qquad \mathbb{R}\oplus\mathbb{R}$$

image has dim 1

$$\to H^1(U_1)\oplus H^1(V_1) \qquad \underline{\underline{\text{so}}} \qquad H^1(U_1\cup V_1)=\mathbb{R}$$
$$0 \qquad 0$$

Putting all this together

$$H^1(S^2,\mathbb{R})=0 \qquad H^2(S^2,\mathbb{R}) \cong \mathbb{R}$$