Summary of Things to Know for Math 225B Final

Basic definitions and calculations for manifolds and local coordinate changes including tangent vectors

$\frac{\partial}{\partial \hat{x}_i} = \sum_{j=1}^{n} \frac{\partial x_j}{\partial \hat{x}_i} \frac{\partial}{\partial x_j}$. F_* $(= dF)$ if $F: M_1 \to M_2$: definition and local coordinate calculations

Definition of vector fields, Lie bracket, proof Lie bracket is vector field

Integral curves of vector fields: small-time existence and uniqueness (differential equations result)

Flow of vector fields, one-parameter group of diffeomorphisms

Basic examples: infinitesimal generator of rotations, translations,

Fundamental Theorem: $[X,Y] \equiv 0 \iff$ flows of X, Y commute

Proof via interpretation of $[X,Y]$ as "Lie derivative" of Y with respect to (flow of) X.

Important trick: $X(p) \neq 0 \in T_pM \Rightarrow X \equiv \partial/\partial x_1$ some coordinate system in nbhd of p. This gives $[X,Y]$ = Lie derivative where $X \neq 0$ (general result then follows).

Application: Frobenius theorem: Distribution generated by $X_1, \ldots, X_k$ which is "involutive" ($[X_i, X_j] \in \text{span}(X_1, \ldots, X_k)$) has integral submanifolds.

Example: When is there (locally) a function f such that fV has curl $\equiv 0$, given a vector field V nowhere $= \vec{0}$.

(approach via: happens if $V^\perp$ is integrable ($\iff$ involutive))

Results from homework on 1-parameter subgroups of Lie groups.

Example: $\gamma(t)$ one parameter subgroup of $Gl(n, \mathbb{R})$ lies in $SO(n) \iff \left.\frac{d\,\gamma(t)}{dt}\right|_{t=0}$ is skew symmetric)

Differential forms: T_pM^* dual space of T_pM. Basis $dx_1, \ldots, dx_n$

differential 1-form $p \to T_pM^*$ C^∞ in obvious sense. Locally $\sum f_j\, dx_j$. Change of coordinates $d\hat{x}_j = \sum_{\ell} \frac{\partial \hat{x}_j}{\partial x_\ell} dx_\ell$

Differential of function f: $df(v) = vf$, $v \in T_pM$

Things to know ② for 225B final

$F: M_1 \to M_2$, ω 1-form on M_2, $F^*\omega$ is 1-form on M_1 defined by $(F^*\omega)(v)$
$= \omega|_{F(p)}(F_* v)$, $v \in T_p M_1$.

Differential forms "pull back" even though vector fields do not push forward. (Vectors push forward, but not vector fields)

k-forms: at each point p, antisymmetric multilinear map
$T_pM \times \dots \times T_pM$ (k factors) $\to \mathbb{R}$.

Basis: $dx_{i_1} \wedge \dots \wedge dx_{i_k}$, $i_1 < \dots < i_k$, dimension (at each point) $\binom{n}{k}$ $\dim M = n$

→ $\wedge$ product definition.

Important operator: d: k forms → $(k+1)$ forms.

<u>Not</u> pointwise: involves differentiation.

$$d(f\, dx_{i_1} \wedge \dots \wedge dx_{i_k}) = df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$$

Coordinate invariance, coordinate invariant formula:

$$(d\omega)(X_0, \dots, X_k) = \sum (-1)^j X_j\, \omega(X_0, \dots, X_k) - \sum \pm\, \omega([X_i, X_j], X_0 \dots X_k)$$

X_j omitted; X_i, X_j omitted; $\pm = (-1)^{i+j+1}$

Important idea:
"pointwise" same as function linear.

Check RHS is function linear. Then can use as definition of $d\omega$ if desired. Example ω 1-form

$$(d\omega)(X_0, X_1) = X_0\, \omega(X_1) - X_1\, \omega(X_0) - \omega([X_0, X_1]).$$

Check function linearity of RHS.

Leibnitz Rule for d: $d(\omega \wedge \theta) = d\omega \wedge \theta + (-1)^{\deg \omega}\, \omega \wedge d\theta$
(proof: compute in local coordinates)

Fundamental facts: (1) "d commutes with pullbacks": $F: M_1 \to M_2$
ω k form on M_2 then $d_{M_1}(F^*\omega) = F^*(d_{M_2}\omega)$.

(2) "$d^2 = 0$": $d_{k+1}(d_k \omega) \equiv 0$, ω a k-form.

Proof of (1): Commutes with wedge products so enough to check (from Leibnitz) on functions. of (2): check in local coordinates.

From (2): image of d on k-1 forms $\subset$ kernel of d on k forms.

Important item: "deRham k-cohomology" notation $H^k_{deR}(M, \mathbb{R})$
$\overset{=}{\text{def}}$ (ker d on k-forms) / (image of d on (k-1)-forms)

Generally quotient of ∞ dim'l vector spaces but if M compact, quotient is finite dimensional (proof later)

d commutes with pullback ⇒ $F: M_1 \to M_2$ induce $F^*: H^k_{deR}(M_2, \mathbb{R}) \to H^k_{deR}(M_1, \mathbb{R})$

Important example: $M =$ Compact genus g surface
$H^1_{deR}(M,\mathbb{R}) = \mathbb{R}^{2g}$. Proof using: simply connected $\Rightarrow H^1_{deR} = 0$ and integral conditions for closed being exact
(simple connectivity result uses idea of integrating closed form along continuous (not nec. diff.) curve and corresponding concept for homotopy).

Poincare Lemma (actually proved by Volterra): M smoothly contractible $\Rightarrow H^k_{deR}(M,\mathbb{R}) = 0$, $k \geq 1$.
Proof using $\mathcal{L}_X = d i_X + i_X d$ formula. ← Know proof & how linear contraction ideas work in practice

deRham cohomology for S^n, $\mathbb{RP}^n$ (n odd versus n even). [Homework]

Riemannian metrics: pos. def. sym. bilinear form on each T_pM. $\langle , \rangle|_p$ or $g|_p$
Defines idea of lengths of piecewise smooth curves $\gamma : [a,b] \to M$ $\quad \ell(\gamma) = \int_a^b \langle \gamma'(t), \gamma'(t) \rangle^{1/2}|_{\gamma(t)}\, dt$

$$= \int_a^b \| \gamma'(t) \| \big|_{\gamma(t)}\, dt.$$

$\mathrm{dist}(p,q) = \inf\limits_{\gamma \text{ from } p \text{ to } q} \ell(\gamma)$. Defines metric space structure (metric space top. = manifold topology)
inf not always realized ($\mathbb{R}^2 - \{(0,0)\}$)
M complete in metric space structure $\Rightarrow$ distance realized (not proved)

Differentiation of vector fields: coordinate idea (diff. coefficients) is not coordinate invariant. But given Riemannian metric, get a differentiation operator by using formula derived from two basic properties of $\mathbb{R}^n$ derivative

(1) $D_X Y - D_Y X = [X,Y]$ & (2) $X\langle Y,Z\rangle = \langle D_X Y, Z\rangle + \langle Y, D_X Z\rangle$

Get: $2\langle D_X Y, Z\rangle = X\langle Y,Z\rangle + Y\langle X,Z\rangle - Z\langle X,Y\rangle$
$\qquad + \langle [X,Y], Z\rangle - \langle [Y,Z], X\rangle - \langle [X,Z], Y\rangle$

Use this for definition: check RHS func linear in X, Z, Leibnitzian in Y so definition makes sense.

Fact: Geodesics (curves with acc $= 0$) are locally minimal connections

Also minimal connections are geodesics (up to parameterization)

Basic idea of geodesic Riemannian geometry: (M, g) complete $\Rightarrow$ with p given, union of geodesics from $p = M$ (all geodesics ∞ly extendible) so control topology of M by understanding geodesics in terms of "curvature" (generalization to all dimensions of Gauss curvature of surfaces in $\mathbb{R}^3$).

Integration and Stokes' Theorem: definition of integration of n-form (of compact support) on oriented n-manifold (with or without boundary): convert to local coordinate question by partition of unity. Proof of coordinate cover invariance by change of variables for multiple integrals together with $(\Sigma \rho_\lambda)(\Sigma \alpha_\mu) = \sum_{\lambda,\mu} \rho_\lambda \alpha_\mu$ (ρ, α part. of unity).

Stokes Theorem: $\int_M d\omega = \int_{\partial M} \omega$, ω $n-1$ form, M manifold with boundary, oriented. (∂M is given induced orientation)

Proof via partition of unity and iterated integration.

Riemannian metric and differential forms

M with Riemannian metric g, $v_1, \dots, v_n$ g-orthonormal on T_pM, $\theta_1, \dots, \theta_n$ dual basis of T_pM^*. Define $\theta_{i_1} \wedge \dots \wedge \theta_{i_k}$, $i_1 < \dots < i_k$ to be orthonormal for $\Lambda^k T_pM^*$ (k-forms at p).

Proof that this is independent of choice of ortho. frame $v_1, \dots, v_n$ by using 1-parameter subgroup reasoning to show $SO(n)$ acting on $v_1 \dots v_n$ induces a $SO(\binom{n}{k})$ action on $\Lambda^k T_pM^*$-basis $\theta_{i_1} \wedge \dots \wedge \theta_{i_k}$ (problem is reduced to linear situation $\frac{d\gamma(t)}{dt}\big|_{t=0}$ being skew-symmetric hence to rotation in θ_i, θ_j only, which case is easily checked).

Definition of $*$ operator ($*(\theta_{i_1} \wedge \dots \wedge \theta_{i_k}) = \pm \wedge$ of omitted θ_j with $\pm$ determined so $\alpha \wedge * \alpha$ is positively oriented (M is oriented here!)

Ideas of harmonic theory: Look for distinguished representative in each deRham k-cohomology class by trying to minimize $\langle\langle \omega, \omega \rangle\rangle \stackrel{\text{def}}{=} \int_M \langle \omega, \omega \rangle$ ($\langle\, , \rangle$ is pointwise inner prod. from orthonormal basis choice as above)

Look for $d\omega = 0$, $\delta\omega = 0$. Stokes $\int d(\alpha \wedge * \beta) = 0$ gives $\delta = \pm * d *$. where δ = adjoint of d.

So look for "harmonic" $\omega \ni d\omega = 0$ $d(*\omega) = 0$. Hodge Theorem: Each deRham class contains unique harmonic form. Cor: Poincare duality $H^k_{deR}(M, \mathbb{R}) \cong H^{n-k}_{deR}(M, \mathbb{R})$