Sample Problems 225B Final

(1) Prove by calculation: If $f(x,y)\,dx + g(x,y)\,dy = h(r,\theta)\,dr + k(r,\theta)\,d\theta$ (on $\mathbb{R}^2$), then

$$\left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \left(\frac{\partial k}{\partial r} - \frac{\partial h}{\partial \theta}\right)(dr \wedge d\theta)$$

(i.e. d in x,y coordinates $= d$ in (r,θ) coordinates)

(2) Explain carefully how the Divergence Theorem for the Unit Ball in $\mathbb{R}^3$ Follows From Stokes Theorem for Manifolds-With-Boundary

(3) Prove that if ω is a C^∞ 1-form on $S^2 \ni d\omega \equiv 0$, then $\exists\ C^\infty\ F: S^2 \to \mathbb{R} \ni dF \equiv \omega$.

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4) Let ω be the 2-form on $\mathbb{R}^3$ (in standard coordinates x,y,z) with $(0,0,0)$ removed:

$$\frac{x}{(x^2+y^2+z^2)^{3/2}}\,dy\wedge dz - \frac{y}{(x^2+y^2+z^2)^{3/2}}\,dx\wedge dz + \frac{z}{(x^2+y^2+z^2)^{3/2}}\,dx\wedge dy$$

(a) Prove that $d\omega \equiv 0$ on $\mathbb{R}^3 - \{(0,0,0)\}$

(b) Evaluate $\int_{\text{unit sphere}} \omega$, unit sphere $= \{(x,y,z): x^2+y^2+z^2=1\}$, exterior (outward) normal orientation

(c) Evaluate $\int_S \omega$, where $S = \{(x,y,z): x^2+3y^2+4z^2 = 100\}$ outward normal orientation.

[Suggestion: For (c) and even (b), think about Stokes' Theorem]

(5) (a) Prove carefully that $F: \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 0$ if $x < 0$, $F(x) = e^{-1/x^2}$ if $x > 0$ is C^∞.

(b) Use part (a) to find a function $p: \mathbb{R} \to \mathbb{R} \ni$ $p = 0$ on $\mathbb{R} - (-2, 2)$, $p \geq 0$ everywhere, and $p \equiv 1$ on $[-1, 1]$.

(6) Describe in detail how to construct some nonorientable manifold (your choice) and prove carefully that the manifold you construct is in fact nonorientable.

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(7) Prove: ω 2 form on S^2 then $\exists$ 1 form θ $\ni$ $\omega = d\theta \iff \iint_{S^2} \omega = 0$.

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(8) Describe how the Poincare Lemma proof construction produces a 1-form θ on $\mathbb{R}^3$ with $d\theta = x\, dy \wedge dz + y\, dx \wedge dz$

(9) Discuss the question of finding a vector field V on $\mathbb{R}^3$ with curl V = a given vector field X with div $X = 0$ in terms of differential forms and Poincare Lemma.

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(10) Show how Stokes' Theorem for oriented surfaces with boundary (in $\mathbb{R}^3$) follows from Stokes' Theorem for differential forms (classical Stokes involves $\iint \text{curl}\, V \cdot N \, d(\text{area})$ etc.)

(11) (a) Explain how a closed 1-form gives rise to a Čech 1-cocycle for a cover $\{U_i : i \in \Lambda\}$ of a manifold M if the cover has the right properties (and say what these properties are.

(b) Show that your map in part (a) induces a map H^1_{deRham} to H^1_{Cech}.

(12) Explain how to use a partition of unity to invert the map you found in 11(b).

(13) (a) Define the $*$ operator on differential forms on a manifold M with Riemannian metric g.

(b) Show that $** = \pm 1$, $\pm$ depending only on $\dim M$ and degree of form involved.

(14) (a) State the Hodge Theorem on harmonic representatives of deRham cohomology classes.

(b) Suppose M is compact, oriented. Use part (a) to show that $H^n_{deR}(M, \mathbb{R}) = \mathbb{R}$, $n = \dim M$

(15) Use the Hodge Theorem to show (using 13(b)) that

$$H^k_{deR}(M, \mathbb{R}) \cong H^{n-k}_{deR}(M, \mathbb{R}) \qquad n = \dim M, M \text{ compact oriented}$$

(16) Prove that if M is orientable, $\exists$ an n-form on M, $n = \dim M$, which is nowhere zero.

(17) Prove the converse of (16) ($\exists$ nowhere zero form $\Rightarrow$ orientable)

(18) What is the deRham cohomology of S^n? Prove your answer inductively.

(19) Show that $H^k_{deR}(\mathbb{RP}^n, \mathbb{R}) = 0$ if $1 \le k < n$.

(20) Determine $H^n_{deR}(\mathbb{RP}^n, \mathbb{R})$ and prove your answer. (you may assume anything true about S^n)