The Poincaré Lemma (actually proved originally by Volterra): If M is smoothly contractible, then $d\omega = 0$ on $M \Rightarrow \exists\, \theta$ on M with $\omega = d\theta$.

Actually, we shall prove more: If $F_t : M \to N$ is a C^∞ family of C^∞ maps, $t \in [0,1]$ and if ω is a closed k-form on N, $k \geq 1$, then $\exists\, \Theta$ on M such that

$$F_1^*\omega - F_0^*\omega = d\theta. \qquad (*)$$

To get the statement of the Poincaré Lemma, let $F_t : M \to M$ be the smooth contraction with $F_1 =$ identity, $F_0 =$ constant map (so $F_0^*\omega = 0$, assuming $\deg\omega \geq 1$).

The proof is to be based on the following Identity: If X is a vector field, then

$$\mathcal{L}_X \Omega = d\, i_X(\Omega) + i_X(d\Omega)$$

where i operation is defined by

$$i\left(\frac{\partial}{\partial x_j}\right) dx_{i_1} \wedge \dots \wedge dx_{i_k} \qquad i_1 < \dots < i_k,\ k = \deg\Omega,$$

dx_j deleted

$$= (\pm 1)\, dx_{i_1} \wedge \dots \widehat{\ } \dots \wedge dx_{i_k} \text{ if } j \in \{i_1, \dots i_k\},\ 0 \text{ otherwise},$$

and ± 1 is determined by how many dx_i's one has to move past to get to j, so

$$i_{\left(\frac{\partial}{\partial x_j}\right)}\, dx_j \wedge dx_{i_2} \dots = dx_{i_2} \wedge dx_{i_3} \wedge \dots$$

$$i_{\left(\frac{\partial}{\partial x_j}\right)}\, dx_{i_1} \wedge dx_j \wedge dx_{i_3} \wedge \dots = -\, dx_{i_1} \wedge dx_{i_3} \dots$$

etc.

Proof of the formula: By our usual simplification it is enough to check the case $X = \partial/\partial x_1$ in some coordinate and $\Omega = f\, dx_{i_1} \wedge \ldots \wedge dx_{i_k}$. $i_1 < \ldots < i_k$. There are two cases

(1) $1 \notin \{i_1, \ldots, i_k\}$

(2) $1 \in \{i_1, \ldots, i_k\}$ so $i_1 = 1$

Case 1: $i_X \Omega = 0$ so $(d i_X + i_X d)\Omega$

$$= i_X d\Omega = \sum_{l,i} i_{\partial/\partial x_1} \frac{\partial f}{\partial x_l} dx_l \wedge dx_{i_1} \wedge \ldots \wedge dx_{i_k}$$

$$= \sum_i \frac{\partial f}{\partial x_1} dx_{i_1} \wedge \ldots \wedge dx_{i_k} = \mathcal{L}_X \Omega \left(= \mathcal{L}_{\frac{\partial}{\partial x_1}} \Omega\right)$$

Case 2: $\Omega = f\, dx_1 \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$

so

$$d\left(i_{\partial/\partial x_i} \Omega\right) = d\left(f\, dx_{i_2} \wedge \ldots \wedge dx_{i_k}\right)$$

$$= \sum_{l=1}^{n} \frac{\partial f}{\partial x_l} dx_l \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$$

while

$$i_{\frac{\partial}{\partial x_1}} d\Omega = i_{\partial/\partial x_1}\Big(\sum_{l \neq 1} \frac{\partial f}{\partial x_l} dx_l \wedge \underset{\overset{\|}{1}}{dx_{i_1}} \wedge \ldots dx_{i_k}\Big)$$

$$= -\sum_{l \neq 1} \frac{\partial f}{\partial x_l} dx_l \wedge dx_{i_2} \wedge \ldots dx_{i_k}$$

So $(d i_{\partial/\partial x_1} + i_{\partial/\partial x_i} d)\Omega = \dfrac{\partial f}{\partial x_1} dx_1 \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}$

$$= \mathcal{L}_{\partial/\partial x_1}(f\, dx_1 \wedge dx_{i_2} \wedge \ldots \wedge dx_{i_k}) = \mathcal{L}_{\partial/\partial x_1} \Omega.$$

Now we return to the $F_1^* \omega - F_0^* \omega$ is exact situation: First, we set $\Omega =$ the k-form on $M \times I$ obtained by pulling-back ω on N via the map $G . M \times I \to N$ defined by $G(x,t) = F_t(x)$. Then we take $X = \partial/\partial t$ in our $\mathcal{L}_X \omega$ formula.

$\mathcal{L}_X$ is just differentiation of coordinates as a function of t in the sense that if

$$\Omega = \sum_{i_1 < \dots < i_k} f(x,t)\, dx_{i_1} \wedge \dots \wedge dx_{i_k} + \sum_{\substack{J \\ j_1 < \dots < j_{k-1}}} g(x,t)\, dt \wedge dx_{j_1} \wedge \dots dx_{j_{k-1}}$$

then

$$\mathcal{L}_X \Omega = \sum \frac{\partial f}{\partial t} dx_{i_1} \dots \wedge dx_{i_k} + \sum \frac{\partial g}{\partial t}\, dt \wedge dx_{j_1} \wedge \dots dx_{j_{k-1}}$$

Now $F_0^*(x)\omega$ as a form on M and $F_1^*(x)\omega$ as forms on M are obtained from Ω by dropping the terms containing dt. It follows that (as forms on M)

$$F_1^* \omega - F_0^* \omega = \int_0^1 F_t^*(\mathcal{L}_X \Omega)\, dt = \int_0^1 F_t^*(\mathcal{L}_{\partial/\partial t} \Omega)\, dt$$

where $F_t^* : M \to M \times I$ is defined as in our previous notation. Note that F_t^* simply erases the terms involving g's.

But $\mathcal{L}_X \Omega = d(i_X \Omega) + i_X(d\Omega) = d(i_X \Omega)$ since $d\Omega = 0$ (Ω is the pull back of the closed form ω on M). i.e. $\mathcal{L}_{\frac{\partial}{\partial t}} \Omega = d_{M \times I}(i_{\partial/\partial t} \Omega)$

Integrating gives

$$F_1^* \omega - F_0^* \omega = \int_0^1 F_t^*(\mathcal{L}_X \Omega)$$

$$= \int_0^1 F_t^* \left(d_{M \times I} (i_{\frac{\partial}{\partial t}} \Omega) \right) = \int_0^1 d_M \left(F_t^* (i_{\frac{\partial}{\partial t}} \Omega) \right) dt$$

integration symbol, not form!

$$= d_M \left(\int_0^1 F_t^* (i_{\partial/\partial t} \Omega)\, dt \right).$$

The notation confuses a bit, so let us work out some explicit examples.

$M = N$:
First, consider the 1-form $\omega = f(x)\,dx$ on $\mathbb{R}$ with $F: \mathbb{R} \times [0,1] \to \mathbb{R}$ defined by $F(x,t) = tx$.
Then F_0 is the map taking all of $\mathbb{R}$ to 0.
So $F_0^*(f(x)\,dx) = 0$ while $F_1^*(f(x)\,dx) = 0$.
Thus, $(F_1^* - F_0^*)(\omega)$ is exact will be simply the statement that ω is exact ($d\omega = 0$ on $\mathbb{R}$ since no 2-form on $\mathbb{R}$ is anything but $\equiv 0$!).
Now Ω on $M \times I$ is determined by

$$\Omega\big|_{(x,t)}\left(\frac{\partial}{\partial x}\right) = f(\cancel{x})\,d\cancel{x}\big|_{tx}\left(F_*\frac{\partial}{\partial x}\Big|_{(x,t)}\right)$$

$$= t\,f(tx) \quad \text{since } F_*\frac{\partial}{\partial x}\Big|_{(x,t)} = \frac{d}{d\cancel{x}}(tx) = t\frac{\partial}{\partial x}$$

$$\Omega\big|_{(x,t)}\left(\frac{\partial}{\partial t}\right) = f(\cancel{x})\,d\cancel{x}\big|_{tx}\;F_*\left(\frac{\partial}{\partial t}\right)$$

$$= f(tx)\cdot\frac{d}{dt}(tx) = x\,f(tx)$$

So $$d\left(i_{\partial/\partial t}\,\Omega\right) = d\left(\Omega\left(\frac{\partial}{\partial t}\right)\right) = x\,f(tx).$$

Thus $(F_1^* - F_0^*)\,\omega$ should be $\displaystyle\int_0^1 x\,f(tx)\,dt$

$= \mathcal{F}(x) - F(0)$ if $\mathcal{F}$ satisfies $\dfrac{d\mathcal{F}(s)}{ds} = f(s)$

(this is because

$$\frac{\partial}{\partial t}f(tx) = x \quad \text{so} \quad \int_0^1 x\,f(tx) = \int_0^1 x\,F'(tx)\,dx = \mathcal{F}(tx)\Big|_0^1 = \mathcal{F}(x) - \mathcal{F}(0)).$$

So
$$d\left(\int_0^1 x\,f(tx)\,dt\right) = d\mathcal{F} = \frac{dG}{dx}\,dx = f(x)\,dx$$
as required. An odd way to integrate, but it works!

Here is another example $M = N = \mathbb{R}^2$, $\omega = x\,dy + y\,dx$,

$F_t(v) = tv$. $\quad (d\omega = dx \wedge dy + dy \wedge dx = 0)$

$F_t(x,y) = (tx, ty)$ $\quad \left(F_t^*\omega\right)\left(\frac{\partial}{\partial x}\right)\Big|_{(x,y,t)} = (x\,dy + y\,dx)\Big|_{F(x,y,t)} \cdot F_* \frac{\partial}{\partial x}$

$$= t \cdot ty = t^2 y$$

since $F_* \frac{\partial}{\partial x} = \frac{\partial}{\partial x}(xt, yt) = t(1,0) = t\frac{\partial}{\partial x}$

We DO not really use this part.

while $\left(F_t^*\omega\right)\left(\frac{\partial}{\partial y}\right) = (x\,dy + y\,dx)\Big|_{F(x,y,t)}\left(F_* \frac{\partial}{\partial y}\right)$

$$= t^2 x$$

and $\left(F^*\omega\right)\left(\frac{\partial}{\partial t}\right) = (x\,dy + y\,dx)\Big|_{F(x,y,t)}\left(F_* \frac{\partial}{\partial t}\right)$ $\leftarrow = (x,y)$

$$= (ty)\cdot x + tx\,dy\left(F_* \frac{\partial}{\partial t}\right)$$

$dx\left(F_* \frac{\partial}{\partial t}\right)$

$$= 2xyt$$

Use this part

So

$F_1^*\omega - F_0^*\omega$ should be $d\left(\int_0^1 2xyt\,dt\right)$

$$= d\left(2xy\,t^2/2\right)\Big]_0^1 = d(xy)$$

This works! $\quad d(xy) = x\,dy + y\,dx$!

The combinatorics here are odd at first sight.
But it does work.

Now we try a two-form on $\mathbb{R}^3$: $\omega = x\,dy\wedge dz + y\,dx\wedge dz$.

$d\omega = dx\wedge dy\wedge dz + dy\wedge dx\wedge dz = 0$. Let $M = N = \mathbb{R}^3$,

$F_t((x,y,z)) = (tx, ty, tz)$. So

$$(F_t)_*\frac{\partial}{\partial t} = \frac{d}{dt}(tx, ty, tz) = (x,y,z)$$

$$(F_t)_*\left(\frac{\partial}{\partial x}\right) = \frac{d}{dx}(tx, ty, tz) = t\frac{\partial}{\partial x}$$

$$(F_t)_*\left(\frac{\partial}{\partial y}\right) = t\frac{\partial}{\partial y}$$

$$(F_t)_*\left(\frac{\partial}{\partial z}\right) = t\frac{\partial}{\partial z}$$

$$(F_t^*\omega)\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \omega((x,y,z),(t,0,0))\Big|_{F_t(x,y,z)} = -t^2yz$$

(only $dx\wedge dz$ term contributes $-tz\cdot\cancel{t}\cdot$ coef $\cancel{y} = -tz\cdot ty = -t^2yz$)

$$(F_t^*\omega)\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial y}\right) = \omega((x,y,z),(0,t,0))$$

$$= -tz\cdot\text{coef } x\Big|_{(xt,yt,zt)}$$

$$= -t^2xz$$

$$(F_t^*\omega)\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial z}\right) = \omega((x,y,z)\ (0,0,t))\Big|_{(tx\ ty\ tz)}$$

$$= tx\cdot ty + ty\cdot tx = 2t^2xy$$

So $\displaystyle\int_0^1 i_{\frac{\partial}{\partial t}} F_t^*\omega = \int_0^1 -t^2yz\,dx - t^2xz\,dy + 2t^2xy\,dz$

$$= -\tfrac{1}{3}yz\,dx - \tfrac{1}{3}xz\,dy + \tfrac{2}{3}xy\,dz$$

$d(\cdot) = \omega$? $\quad d(\ ) = -\cancel{\tfrac{1}{3}z\,dy\wedge dx} - \tfrac{1}{3}y\,dz\wedge dx$

$$-\cancel{\tfrac{1}{3}z\,dx\wedge dy} - \tfrac{1}{3}x\,dz\wedge dy$$

$$+\tfrac{2}{3}y\,dx\wedge dz + \tfrac{2}{3}x\,dy\wedge dz$$

$$= y\,dx\wedge dz + x\,dy\wedge dz \checkmark$$

$= \omega$ ✓ as expected.