More Frobenius Observations

Specific case of proof idea: X, Y vector fields on $\mathbb{R}^3$

$X = f_1\frac{\partial}{\partial x} + f_2\frac{\partial}{\partial y} + f_3\frac{\partial}{\partial z}$, $Y = g_1\frac{\partial}{\partial x} + g_2\frac{\partial}{\partial y} + g_3\frac{\partial}{\partial z}$

Suppose at $(0,0,0)$ $\quad f_1 g_2 - g_1 f_2 \neq 0$ (wolog)

Then

$$V = \frac{1}{f_1 g_2 - g_1 f_2}\left(g_2 X - f_2 Y\right) \text{ has form } \frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + (\cdot)\frac{\partial}{\partial z}$$

and

$$W = \frac{1}{f_1 g_2 - g_1 f_2}\left(-g_1 X + f_1 Y\right) \text{ has form } 0\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + (\cdot\cdot)\frac{\partial}{\partial z}$$

If $[X,Y] \in \operatorname{span}(X,Y)$, then $[V,W] \in \operatorname{span}(V,W)$ and then it must be that $[V,W] = \vec{0}$: since $[V,W] = \left(\frac{\partial}{\partial x}(\cdot\cdot) - \frac{\partial}{\partial y}(\cdot)\right)\frac{\partial}{\partial z}$, this can belong to $\operatorname{span}(V,W)$ only if it is zero!
($aV + bW = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + (a(\cdot) + b(\cdot\cdot))\frac{\partial}{\partial z}$ is a general linear combination and this is a multiple of $\frac{\partial}{\partial z}$ only if it is zero, $a=0$, $b=0$).

Another example: $X = \frac{\partial}{\partial x} + F\frac{\partial}{\partial z}$ $\quad Y = \frac{\partial}{\partial y} + G\frac{\partial}{\partial z}$

$[X,Y] \in \operatorname{span}(X,Y) \implies [X,Y] = 0 \iff$

$$-\frac{\partial F}{\partial y} + \frac{\partial G}{\partial x} + F\frac{\partial G}{\partial z} - G\frac{\partial F}{\partial z} = 0. \quad (*)$$

A surface $z = f(x,y)$ has tangent space = span of X, Y if and only if $\operatorname{grad}(f(x,y) - z) \perp X$ and Y or $\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, -1\right) \perp X$ and $Y \iff$

$\frac{\partial f}{\partial x} - F = 0$ & $\frac{\partial f}{\partial y} - G = 0$. For this, we need

in trying to find f
(on account of equality of mixed partials)

$$\frac{\partial}{\partial y}\left(F(x,y,f(x,y)) \right) = \frac{\partial}{\partial x}\left(G(x,y,f(x,y)) \right)$$

or $$\frac{\partial F}{\partial y} + \frac{\partial f}{\partial y}\cdot\frac{\partial F}{\partial z} = \frac{\partial G}{\partial x} + \frac{\partial f}{\partial x}\frac{\partial G}{\partial z}$$

or $$\frac{\partial F}{\partial y} + G\frac{\partial F}{\partial z} = \frac{\partial G}{\partial x} + F\frac{\partial G}{\partial z}$$ or

$$-\frac{\partial F}{\partial y} + \frac{\partial G}{\partial x} - G\frac{\partial F}{\partial z} + F\frac{\partial G}{\partial z} = 0.$$

This is the same as the Lie bracket condition we found earlier! (equation (*)).

Thus we see how the Lie bracket condition arises in this concrete calculus situation.