# Lie Derivatives of Differential Forms.

Several equivalent ideas:

(1) $X$ vector field, $\varphi_t$ local flow, $\omega$ differential form (k-form)

$$\mathcal{L}_X \omega = \lim_{t \to 0^+} \frac{\varphi_t^* \omega - \omega}{t}$$

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$$(2) \; (\mathcal{L}_X \omega)(Y_1, \ldots, Y_k) = X\,\omega(Y_1, \ldots, Y_k) - \sum_{i=1}^{k} \omega(Y_1, \ldots [X, Y_i], \ldots Y_k)$$

Note: $\underline{\underline{no}}$ $\pm$, $(-1)^i$ here

(3) Suffices to define $\mathcal{L}_X \omega$ at p in case $X(p) \neq 0$

(usual trick: p where $X \equiv 0$ in a neighborhood, set $\mathcal{L}_X \omega|_p = 0$, p where $X(p) = 0$ but $X \not\equiv 0$ in nbhd, define $\mathcal{L}_X \omega$ by limit of $X \neq 0$ case).

If $X = \partial/\partial x_1$, then write $\omega = \sum_{i_1 < \ldots < i_k} f_{i_1 \ldots i_k} dx_{i_1} \wedge \ldots \wedge dx_{i_k}$

and set $$\mathcal{L}_X \omega = \sum_{i_1 < \ldots < i_k} \frac{\partial f_{i_1 \ldots i_k}}{\partial x_1} dx_{i_1} \wedge \ldots \wedge dx_{i_k}.$$

Exercises: (a) RHS in (2) is function-linear in $Y_1, \ldots, Y_k$ and hence eq (2) defines $\mathcal{L}_X \omega$ as a form.

(b) Definition in (3) agrees with definition (2).

Since (3) and (1) clearly agree in case $X(p) \neq 0$, it follows that all three ideas of $\mathcal{L}_X \omega$ coincide (in all cases, by observations in (3)).

This is worth thinking over! It is also worth working out some specific examples. One of these examples is provided on the next two pages: polar coordinate things, as we often do.

Concrete Example of Lie Derivatives: Coordinate Version versus Differentiation along Flows Version versus Algebra Version:

In polar coordinates: $x = r\cos\theta,\ y = r\sin\theta$

$\mathcal{L}_{\frac{\partial}{\partial\theta}} dx$ on $\mathbb{R}^2$: $dx = \frac{\partial x}{\partial r}\, dr + \frac{\partial x}{\partial \theta}\, d\theta$

$$= \cos\theta\, dr + (-r\sin\theta)\, d\theta$$

So $\mathcal{L}_{\frac{\partial}{\partial\theta}} dx = -\sin\theta\, dr - r\cos\theta\, d\theta = -dy$ (see footnote)

Also recall $\frac{\partial}{\partial\theta} = \frac{\partial x}{\partial\theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial\theta}\frac{\partial}{\partial y} = -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}$

Now we can check all this versus $(\mathcal{L}_X\omega)Y = X(\omega(Y)) - \omega([X,Y])$

$$\Big(\mathcal{L}_{\frac{\partial}{\partial\theta}} dx\Big)\underbrace{\frac{\partial}{\partial x}}_{Y} = \frac{\partial}{\partial\theta}\, dx\Big(\frac{\partial}{\partial x}\Big) - dx\Big(\Big[\frac{\partial}{\partial\theta}, \frac{\partial}{\partial x}\Big]\Big)$$

$$= \frac{\partial}{\partial\theta}(1) - dx\Big(\Big[-r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\Big]\Big)$$

$$= dx\Big(\Big[\frac{\partial}{\partial x}, -r\sin\theta\,\frac{\partial}{\partial x} + r\cos\theta\,\frac{\partial}{\partial y}\Big]\Big)$$

$$= dx\Big(\Big[\frac{\partial}{\partial x}, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big]\Big) = dx\Big(\frac{\partial}{\partial y}\Big) = 0$$

while

$$\Big(\mathcal{L}_{\frac{\partial}{\partial\theta}} dx\Big)\Big(\frac{\partial}{\partial y}\Big) = 0 - dx\Big(\Big[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\Big]\Big)$$

$$= dx\Big(\Big[\frac{\partial}{\partial y}, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\Big]\Big) = dx\Big(-\frac{\partial}{\partial x}\Big)$$

$$= -1$$

So $\mathcal{L}_{\frac{\partial}{\partial\theta}} dx = -dy$. Exercise: Interpret geometrically (in terms of flows).

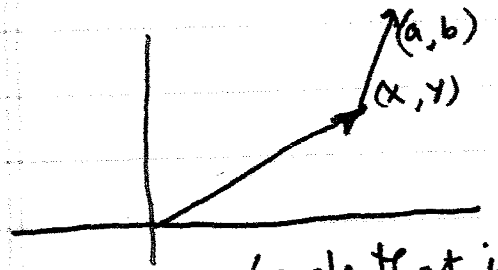
Solution of exercise:

Footnote:

$$-\sin\theta\, dr - r\cos\theta\, d\theta = -\overset{y/r}{\sin\theta}\cdot\Big(\frac{x}{\sqrt{x^2+y^2}}\, dx + \frac{y}{\sqrt{x^2+y^2}}\, dy\Big)$$
$$- \underbrace{r\cos\theta}_{x}\Big(\frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy\Big)$$

$$= \Big(\frac{-xy}{x^2+y^2} + \frac{xy}{x^2+y^2}\Big)\, dx$$
$$+ \Big(\frac{-y^2-x^2}{x^2+y^2}\Big)\, dy = -dy$$

We recall that the flow of $\frac{\partial}{\partial\theta}$ is rotation at unit rate counter clockwise. Now suppose $v$ is a vector in $T_{(x,y)}\mathbb{R}^2$ $\quad v = (a,b)$




Then $(\varphi_t)_* \Big|_{t=0} v$ is the rotation of $(a,b)$ by angle $t$ (note that it is independent of $(x,y)$ since $\varphi_t$ is linear and hence $(\varphi_t)_*$ is constant in $(x,y)$ but depends on $t$). Now

$$\mathcal{L}_{\frac{\partial}{\partial\theta}} dx \underset{\substack{\uparrow \\ \text{formal definition}}}{=} \lim_{t \to 0^+} \frac{((\varphi_t)^* dx - dx)}{t}$$

Applied to $v$:

$$(\mathcal{L}_{\frac{\partial}{\partial\theta}} dx)v = \lim_{t \to 0^+} \frac{dx((\varphi_t)_* v) - dx(v)}{t}$$

$$= \lim_{t \to 0^+} \frac{\text{x-comp. of angle } t \text{ rotation of } (a,b) - a}{t}$$

$$= \lim_{t \to 0^+} \frac{(\cos t)a - (\sin t)b - a}{t} = -b = -dy\,(a,b)$$

checking that $\mathcal{L}_{\frac{\partial}{\partial\theta}} dx = -dy$.

It all fits together as it should!