When Vector Fields are Coordinate Vector Fields
(details of Jan 20 2010 lecture)

If $(x_1, \ldots x_n)$ is a local coordinate system then at each point q in the coordinate neighborhood, $\frac{\partial}{\partial x_1}\big|_q, \ldots \frac{\partial}{\partial x_n}\big|_q$ are linearly independent in $T_q M$. It is natural to ask when a set of vector fields $X_1, \ldots, X_n$ defined in a neighborhood of a point $p \in M$ and linearly independent at p (i.e., $X_1(p) \ldots, X_n(p)$ are a basis for $T_p M$) in fact arise as $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ for some local coordinate system on a (possibly smaller) neighborhood of p. Note that independence at p implies independence at each point of some suitably chosen neighborhood of p, so the linearly independent (at each point) condition is not a problem.

As it happens, there is an additional obviously necessary condition: Since for any local coordinate system $(x_1, \ldots, x_n)$, the Lie brackets $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right]$ are $\equiv 0$, we must require $[X_i, X_j] \equiv 0$. This turns out to be enough!

Theorem: If $X_1, \dots X_n$ are C^∞ vector fields defined in a neighborhood of $p \in M$, if $X_1(p), \dots X_n(p)$ are a basis for $T_p M$, and if $[X_i, X_j] \equiv 0$ on some neighborhood of p (all $i,j \in \{1, \dots n\}$), then there is a local coordinate system (on some possibly smaller neighborhood of p), say $(x_1, \dots x_n)$ such that $X_i \equiv \partial/\partial x_i$, $\forall$ $i \in \{1, \dots n\}$ on the neighborhood.

Proof: Define $\Phi(x_1, \dots x_n)$

$$= \varphi^n_{x_n}(\varphi^{n-1}_{x_{n-1}}(\dots (\varphi^1_{x_1}(p))$$

where $\varphi^j_\cdot$ is the flow of X_j at time $\cdot$.

Φ is defined for $(x_1, \dots x_n)$ close enough to $(0, \dots 0)$ in $\mathbb{R}^n$. Moreover, Φ is a local diffeomorphism, hence (the inverse of) a local coordinate system around p, by the Inverse Function Theorem:

clearly $d\Phi\left(\frac{\partial}{\partial x_i}\right)\Big|_{(0, \dots 0)} = X_i(p)$

so that $d\Phi$ is an isomorphism at $(0, \dots 0)$ onto $T_p(M)$ (since $X_1(p) \dots X_n(p)$

is a basis for T_pM.

$$\text{Note that} \quad d\Phi\left(\frac{\partial}{\partial x_n}\right)\Big|_{(x_1,..x_n)} = X_n\Big|_{\Phi(x_1,..x_n)}$$

$\mathbb{R}^n \; \frac{\partial}{\partial x_n}$

by definition. But it is not clear that (as we hope) $d\Phi\left(\frac{\partial}{\partial x_j}\right) \equiv X_j$ in the same way. However, if the flows $\phi^1, .. \phi^n$ commute, it follows that $d\Phi\left(\frac{\partial}{\partial x_j}\right) \equiv X_j$ because then

$$\Phi(x_1, \ldots x_n) = \phi^j_{x_j}(\text{The rest of the } \phi\text{s } (p))$$

and hence that $\frac{\partial}{\partial x_j}$ RHS $= X_j$ as desired.

Thus the Theorem is proven if we can show the following

With $X_1, .. X_n$ as described (in particular $[X_i, X_j] \equiv 0$), the flows $\phi_1 .. \phi_n$ of $X_1, .. X_n$ commute.

For this, it is enough to do just two vector fields. In particular it is enough to show (with X, Y vector fields on a nbhd of p)

Lemma: If $X(p) \neq 0$ and $[X, Y] \equiv 0$ in a neighborhood of p, then on a possibly smaller neighborhood the flows of X and Y commute.

This is actually true without the hypothesis that $X(p) \neq 0$ as we shall show later. But this hypothesis makes the proof easier, and it enables us to prove the Theorem: For the Theorem, we know already by hypothesis that none of the X_i's vanish at p since $X_1(p) \dots X_n(p)$ are a basis for T_pM.

To prove the Lemma, we use a sublemma, which is important in its own right:

Sub-Lemma: If $X(p) \neq 0$ (X vector field defined in a neighborhood of p), then there is a local coordinate system around p, $(x_1, \dots x_n)$, such that (on a possibly smaller neighborhood)

$$X \equiv \partial/\partial x_1 .$$

Proof: Choose coordinates $(y_1, \dots, y_n)$ such that $p \leftrightarrow (0, \dots, 0)$. By a linear change, constant coefficients, we can then arrange that $X(p) = \partial/\partial y_1 \big|_p$.

Now define

$$C(x_1, \dots x_n)$$

$= \varphi_{x_1}$ applied to the point with $(y_1, \dots y_n)$ coordinates $= (0, x_2 \dots x_n)$.

where φ_t = the flow of X at time t.
C is then a local diffeomorphism and C^{-1} is the desired local coordinate system. $\square$

Proof of Lemma: If $(x_1, \dots x_n)$ is a local coordinate system with $X \equiv \partial/\partial x_1$ and $Y = \sum_{j=1}^{n} f_j(x_1 \dots x_n) \frac{\partial}{\partial x_j}$

then $[X,Y] \equiv 0 \iff \frac{\partial f_j}{\partial x_1} \equiv 0$ (just compute)

by the hypothesis $[X,Y] \equiv 0$

So $f_j(x_1, \dots x_n) = f_j(x_2, \dots x_n)$ no x_1

on some neighborhood. Now the flow φ of $X \equiv \frac{\partial}{\partial x_1}$ is just, at time t, addition of t, i.e. $\varphi_t(x_1, \dots x_n) = (x_1 + t, x_2 \dots x_n)$

But, since x_1 is not involved in f_j, $j = 1, \dots n$,

with ψ = flow of Y

$\psi_s(x_1+t, x_2, \dots, x_n) = \psi_s(\varphi_t(x_1, \dots x_n))$

$= \psi_s(x_1, \dots x_n)$ with t added to the x_1-coordinate. Thus the flows commute as desired. (This is obvious: the notation makes it look more complicated than it is!)

Thus everything is proved!

The Sublemma gives a convenient interpretation of $[X,Y]$ in general.

If $X(p) = Y(p) = 0$ then $[X,Y]|_p = 0$ (Look at the coordinate formula for $[\ ]$).

If one of $X(p)$ & $Y(p)$, one at least, is $\neq 0$, then the sublemma shows how to interpret $[X,Y]_p$ in terms of coordinate differentiation in a suitable coordinate system.

Later on, we shall give an interpretation that is like coordinate differentiation but does not use coordinates as such, only flows.