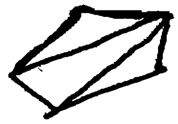
Homework VI

1. Suppose $\{U_\lambda\}$ is a finite open cover. Let c_k = the number of sets $\{\lambda_0, \ldots \lambda_k\}$ of $k+1$ U_λ's with $U_{\lambda_0} \cap \ldots \cap U_{\lambda_k} \neq \phi$.

   Show that
$$\Sigma(-1)^k c_k = \Sigma(-1)^j \dim H^j_{Cech}(\{U_\lambda\})$$
   where $H^j_{Cech}(\{U_\lambda\})$ = the jth Cech cohomology group of the $\{U_\lambda\}$ covering.

2(a) Show that, if a "regular solid" with f faces being k-sided regular polygons has l polygons meeting at each vertex then $f\left(1 - \frac{k}{2} + \frac{k}{l}\right) = 2$.
   (Suggestion: You may assume Euler's formula: see problem 3).

(b) Use part (a) to find the possible values for k, l and f (taken together) and deduce that the five familiar regular solids (tetrahedron, octahedron, cube, dodecahedron, icosahedron) are the only ones there are.

3. Assuming that $f - e + v = 2$ for a triangulation of S^2, show that $f - e + v = 2$ for any decomposition of S^2 into polygons in general (Suggestion  polygons can be subdivided into triangles).

4. Compute that for the tetrahedral cover of S^2 as discussed in class

$$H^2_{Cech} = \mathbb{R} \qquad H^1_{Cech} = 0$$

(This was started in class for H^2).

5. Fill in the following outline to show that if ω is a 2-form on S^2 then

$$\int_{S^2} \omega = 0 \iff \omega = d\theta \text{ for some 1-form } \theta$$

and hence $H^2_{deR}(S^2, \mathbb{R}) \cong \mathbb{R}$:

Steps:

(1) Let $U_1 = \{(x,y,z) \in S^2 : z > -\frac{1}{4}\}$

$U_2 = \{(x,y,z) \in S^2 : z < \frac{1}{4}\}$.

Then $\omega = d\theta_1$ on U_1, $\omega = d\theta_2$ on U_2.

(You may assume the Poincare Lemma)

(2) $$\int_{S^2} \omega = \int_{\substack{\text{equator} \\ \text{west to east}}} \theta_1 - \int_{\substack{\text{equation} \\ \text{west to east}}} \theta_2 .$$

(3) A 1-form θ on $\{(x,y,z) \in S^2 : -\frac{1}{4} < z < \frac{1}{4}\}$ $= dF$ for some function F $\iff$ $\int_{\substack{\text{equation} \\ \text{west to east}}} \theta = 0$

(4) $\int_{S^2} \omega = 0 \Rightarrow \theta_1 - \theta_2 = dF$ on $U_1 \cap U_2$

(5) Required θ on S^2 can be found using θ_1, θ_2, F and a partition of unity ($\theta_1 - d(\rho F)) = \theta_2 + d((1-\rho)F)$ on $U_1 \cap U_2$)