# Homework $\underline{IV}$: Short Course in Geometry of Surfaces ("Do it yourself" course)

Throughout, $S(u,v)$ is a function from an open subset of $\mathbb{R}^2$ into $\mathbb{R}^3$. It is assumed that everywhere $S_* \frac{\partial}{\partial u}$ and $S_* \frac{\partial}{\partial v}$ are linearly independent.

Notation $S_u = S_* \frac{\partial}{\partial u} = \left(\frac{\partial f_1}{\partial u}, \frac{\partial f_2}{\partial u}, \frac{\partial f_3}{\partial u}\right)$ if $S = (f_1, f_2, f_3)$. Similarly $S_{uu} = \left(\frac{\partial^2 f_1}{\partial u^2}, \frac{\partial^2 f_2}{\partial u^2}, \frac{\partial^2 f_3}{\partial u^2}\right)$ etc.

$\partial^2 f_3/\partial u^2$

1. Prove: If $D$ is the Riemannian covariant derivative on $S$ ($S$ considered as a submanifold) then

$$D_X Y = \text{orthogonal projection on tangent space of } S \text{ of } D^{\text{euclidean}}_X Y$$

[Part of the problem is to consider why $D^{\text{euclidean}}_X Y$ makes sense when $X, Y$ are to begin with just defined as tangent vector fields on $S$].

2. Deduce that $$D_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} = S_{uu} - \left\langle \frac{S_u \times S_v}{\|S_u \times S_v\|}, D^{eu}_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} \right\rangle_{eu} \cdot \frac{S_u \times S_v}{\|S_u \times S_v\|}$$

[Note: $\frac{S_u \times S_v}{\|S_u \times S_v\|}$ is a "unit normal" to $S$].

Notation: $$\mathrm{II}(X,Y) = \left\langle \frac{S_u \times S_v}{\|S_u \times S_v\|}, D_X Y \right\rangle$$

3. Prove: $\mathrm{II}(X,Y) = \mathrm{II}(Y,X)$. Conclude that $\mathrm{II}(X,Y)$ is "pointwise" in $X$ and $Y$

4 Discuss why $\langle S_{uu}, S_u\rangle$, $\langle S_{uv}, S_u\rangle$ etc. are "intrinsic", i.e. determined by $\langle S_u, S_v\rangle$, $\langle S_u, S_u\rangle$ and $\langle S_v, S_v\rangle$ and their derivatives. Find formulas for all six cases, e.g. $\langle S_{uu}, S_u\rangle = \frac{1}{2}\frac{\partial}{\partial u}\langle S_u, S_u\rangle$

5. Note that $S_{uvu} = S_{vuu}$. Then compute*

$$\langle S_{vuu}, S_v\rangle = \frac{\partial}{\partial v}\langle S_{uu}, S_v\rangle - \langle S_{uu}, S_{vv}\rangle$$

using $S_{uu} = D_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u} + \mathrm{II}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)\vec{N}$ $\left(\vec{N} = \frac{S_u \times S_v}{\|S_u \times S_v\|}\right)$ etc.

and a similar calculation for

$$\langle S_{uvu}, S_v\rangle = \frac{\partial}{\partial u}\langle S_{vu}, S_v\rangle - \langle S_{vu}, S_{uv}\rangle$$

$$= \frac{\partial}{\partial u}\langle S_{uv}, S_v\rangle - \langle S_{uv}, S_{uv}\rangle$$

to get that $\mathrm{II}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)\mathrm{II}\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) - \mathrm{II}^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)$ is intrinsic. (Suggestion: You will need to use problem 4 to show that $S_{uv} = aS_u + bS_v$ where $a, b$ are intrinsic etc.) Similarly for $S_{uu}$ and $S_{vv}$.

6. Show that

$$\frac{\mathrm{II}\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right)\mathrm{II}\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) - \mathrm{II}^2\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right)}{\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right\rangle\left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right\rangle - \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right\rangle^2}$$

is coordinate independent

[Note: Everything is "pointwise" so this is just linear algebra.

*Crucial point: $D_{\frac{\partial}{\partial u}}\frac{\partial}{\partial u}$ is tangent, so $\perp$ to $\vec{N}$