Homework IV : Poincaré Lemma and deRham stuff

1. Suppose ω is a C^∞ closed k-form on $M \times (-1,1)$. Prove that, if $i: M \to M \times I$ $(= (-1,1))$ is the map $p \to (p, 0)$, then $[\omega] = [\Omega]$ where Ω is the k-form defined on $M \times I$ by sending $(v_1, \dots, v_k)$ to $i_* \omega_p$(tangent to M parts of $v_1, \dots, v_k$), $v_j \in T_{(p,\alpha)}(M \times (-1,1))$ and

$[\ ]$ denotes deRham class in $H^k_{deR}(M \times I, \mathbb{R})$.

In particular, deduce that $\omega = d\theta$ on $M \times I$, some θ if $i_* \omega = d\psi$, some ψ on M.

(Suggestion: Define $H((p,\alpha), t) = (p, \alpha t)$ $t \in [0,1]$ so $H(\cdot, 1) =$ identity on $M \times I$ while $H((p,\alpha), 0)) = p$. Apply Poincaré argument).

2. Use problem 1 to prove inductively that $H^k_{deR}(S^n, \mathbb{R}) = 0$ if $k \geq 1$, $k < n$ by following the outline:

(a) done for $k = 1$. So suppose $k \geq 2$, and true for k's $\leq$ given k.

(b) If ω is a closed k form on S^n, then $\exists\ \theta_1, \theta_2 \ni \omega = d\theta_1$ on $\{(x_1, \dots, x_n) \in S^n : x_n > -\frac{1}{2}\}$ and $\omega = d\theta_2$ on $\{(x_1, \dots, x_n) : x_n < \frac{1}{2}\}$

Where θ_1 is a $k-1$ form on $\{\ x_n > -\frac{1}{2}\} = U$

θ_2 is a $k-1$ form on $\{\ x_n < \frac{1}{2}\} \doteq V$

(c) $\theta_1 - \theta_2$ is a closed $k-1$ form on $U \cap V$ so by induction and problem 1, $\exists\ \psi$ on $U \cap V$ with $\theta_1 - \theta_2 = d\psi$ $\qquad \psi$ $k-2$ form

(d) Use partition of unity-type functions for U, V to modify θ_1, θ_2 by adding $d((\text{func.})\psi)$ so that $d\hat{\theta}_1 = d\theta_1$ on U, $d\hat{\theta}_2 = d\theta_2$ and $\hat{\theta}_1 = \hat{\theta}_2$ on $U \cap V$ so that $\hat{\theta}_1, \hat{\theta}_2$ give desired θ with $d\theta = \omega$.

3. Try to modify your argument in problem 2 to prove (inductively) that $H^n_{dR}(S^n, \mathbb{R}) = \mathbb{R}$ by showing that ω n form on $S^n = d\theta^{n-1}$ if and only if $\int_{S^n} \omega = 0$. (You may assume Stokes Theorem).

4. Use Stokes Theorem to show that $H^n_{deR}(M^n, \mathbb{R}) \neq 0$ if M^n is orientable.

5. Suppose M^n is a compact, (connected) orientable manifold. Assume (as is true) that $H^n_{deR}(U, \mathbb{R}) = 0$ for any open subset U of M that is $\neq M$. Use the ideas from problem 3 to show that if ω is a (automatically closed) n-form on M^n then: ω is exact $\Leftrightarrow \int_M \omega = 0$. Deduce that $H^n_{deR}(M, \mathbb{R}) = \mathbb{R}$.

6. Prove carefully that $\mathbb{RP}^n$ is orientable if n is odd, nonorientable if n is even!

7. Prove that $H^k_{deR}(\mathbb{RP}^n, \mathbb{R}) = 0$ if $1 \le k < n$.

8. Using the result you obtained in problem 3, show that $H^{2n}_{deR}(\mathbb{RP}^{2n}, \mathbb{R}) = 0$, $n = 1, 2, 3 \ldots$

(Suggestion for 7 & 8: A closed k-form ω on $\mathbb{RP}^n$ "pulls back" under $\pi : S^n \to \mathbb{RP}^n$ to a closed form on S^n that is A-invariant, where A = antipodal map. If $\pi^*\omega = d\theta$ then show $\pi^*\omega = d(\frac{1}{2}\theta + \frac{1}{2}A^*\theta)$).