Homework V

1. (Frobenius Theorem, differential forms version)
Suppose $\theta_1, \ldots \theta_k$ are linearly independent differential 1-forms. Define a distribution by $D_p = (n-k)$ plane at p $= \{v \in T_pM : \theta_1(v) = 0, \theta_2(v) = 0, \ldots \theta_k(v) = 0\}$

Show that this distribution is integrable if and only if the 2 forms $d\theta_i$, $i = 1, \ldots k$ have the property that $d\theta_i(v, w) = 0$ for all $v, w \in$ the distribution. (Suggestion: Use $d\omega(X, Y) = X\,\omega(Y) - Y\,\omega(X) - \omega([X, Y])$ to show that this condition is equivalent to the distribution being involutive).

2. With $\theta_1, \ldots \theta_k$ linearly independent 1-forms and D_p as in problem 1, show that a 2-form ω satisfies $\omega(v, w) = 0$ $\forall\, v, w \in T_pM$, $\forall p$, $\mid v, w \in D_p$

if and only if (locally) $\omega = \sum_{i=1}^{k} \theta_i \wedge \omega_i$, some ω_i (Suggestion: Extend $\theta_1, \ldots \theta_k$ to $\theta_1, \ldots, \theta_n$ a basis and note that $\omega = \sum_{i<j} \omega(v_i, v_j)\, \theta_i \wedge \theta_j$

where $v_1, \ldots v_k, v_{k+1}, \ldots v_n$ is the dual basis for T_pM of $\theta_1, \ldots \theta_n$).

3. Suppose M is compact oriented Riemannian. Show that an harmonic n-form ($n = \dim M$) has the form c (vol. form), $c \in \mathbb{R}$ and that every n-form of this sort is harmonic. Deduce that $H^n_{\text{deRham}}(M, \mathbb{R}) \cong \mathbb{R}$.

4. Use problem 3 to show that an n-form ω is exact $\Leftrightarrow \int_M \omega = 0$.

5(a) Suppose that g is a Riemannian metric on an open subset of $\mathbb{R}^2$ ($= \mathbb{C}$) with $g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}\right) = g\left(\frac{\partial}{\partial y}, \frac{\partial}{\partial y}\right)$ and $g\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = 0$. Show that the $*$ operator for g on 1-forms is the same as the $*$ operator for $\mathbb{R}^2$'s usual metric (same orientation of $\mathbb{R}^2$ throughout) namely $*dx = dy$ and $*dy = -dx$.

(b) Deduce that $f_1 dx + f_2 dy$ is harmonic (f_1, f_2 real valued) (relative to g) if and only if

$$\frac{\partial f_1}{\partial y} = \frac{\partial f_2}{\partial x} \quad \text{and} \quad \frac{\partial f_2}{\partial y} = -\frac{\partial f_1}{\partial x}.$$

(c) Deduce that if $u + iv$ is holomorphic then $(u+iv)(dx+idy)$ has real part and imaginary part (after you multiply them out) harmonic. Is the converse true?

6. A Riemann surface is a 2-dimensional manifold M with a coordinate cover $(x_\lambda, y_\lambda) : U_\lambda \to \mathbb{R}^2$ $\lambda \in$ index set Λ such that if $U_\lambda \cap U_\mu \neq \phi$ then, with x_μ and y_μ thought of as functions of (x_λ, y_λ) on $U_\lambda \cap U_\mu$, the function $x_\mu + i y_\mu$ is a holomorphic function of $x_\lambda + i y_\lambda$ (i.e. $\frac{\partial x_\mu}{\partial x_\lambda} = \frac{\partial y_\mu}{\partial y_\lambda}$ and $\frac{\partial x_\mu}{\partial y_\lambda} = -\frac{\partial y_\mu}{\partial x_\lambda}$)

A Riemannian metric g on M is <u>Hermitian</u> (by definition) if $g(\frac{\partial}{\partial x_\lambda}, \frac{\partial}{\partial x_\lambda}) = g(\frac{\partial}{\partial y_\lambda}, \frac{\partial}{\partial y_\lambda})$ and $g(\frac{\partial}{\partial x_\lambda}, \frac{\partial}{\partial y_\lambda}) = 0$

on U_λ, all $\lambda \in \Lambda$.

(a) Prove: If a metric satisfies the Hermitian metric conditions in (x_λ, y_λ) coordinates then it satisfies the conditions in (x_μ, y_μ) coordinates on $U_\lambda \cap U_\mu$ (if (x_μ, y_μ) and (x_λ, y_λ) are holomorphically related as above).

(b) Proof: Every Riemann surface has an Hermitian metric (Suggestion: Partition-of-unity combine the Euclidean metrics $\langle \frac{\partial}{\partial x_\lambda}, \frac{\partial}{\partial x_\lambda} \rangle = \langle \frac{\partial}{\partial y_\lambda}, \frac{\partial}{\partial y_\lambda} \rangle = 1$, $\langle \frac{\partial}{\partial x_\lambda}, \frac{\partial}{\partial y_\lambda} \rangle = 0$ on U_λ's).

(c) If g is an Hermitian metric on a Riemann surface M, then the associated $*$ operator satisfies $* dx_\lambda = dy_\lambda$ and $* dy_\lambda = -dx_\lambda$.

7. Consider d and $*$ extended by complex linearity to complex differential forms ($\alpha\, dx + \beta\, dy$, $\alpha, \beta \in \mathbb{C}$) and complex valued functions.

(a) Show that if $u + iv$ is holomorphic (u, v real valued functions, $\partial u/\partial x_\lambda = \partial v/\partial y_\lambda$, $\partial u/\partial y_\lambda = -\partial v/\partial x_\lambda$) then $(u+iv)\, dz_\lambda$ ($dz_\lambda = dx_\lambda + i\, dy_\lambda$) is harmonic in the sense that $d = 0$ and $d* = 0$.

(b) Show that a complex 1-form that is given as (holomorphic function) dz_λ in every (x_λ, y_λ) coordinate system = in μ coordinates (another holomorphic function) dz_μ on $U_\lambda \cap U_\mu$.

[A complex 1-form expressible as given is called a holomorphic 1-form. So part (a) can be expressed: holomorphic 1-forms are harmonic.]

(c) The Riemann sphere $\mathbb{C} \cup \{\infty\}$ is the Riemann surface with coordinate cover $U_1 = \mathbb{C}$, $U_2 = (\mathbb{C} - \{0\}) \cup \{\infty\}$ with maps $U_1 \to \mathbb{R}^2$ the map taking $x+iy$ to (x, y) and $U_2 \to \mathbb{R}^2$ being the map taking $x + iy \to \left(\mathrm{Re}\left(\frac{1}{x+iy}\right), \mathrm{Im}\left(\frac{1}{x+iy}\right)\right)$ and $\infty \to 0$.

Check this actually is a Riemann surface

* (d) Show that there are no $\not\equiv 0$ holomorphic 1-forms on the Riemann sphere two ways:

(1) Hodge theorem

(2) $dz_2 = -\frac{1}{z_1^2}\, dz_1$ (z_1 on U_1, z_2 coord on U_2)

So $f_2\, dz_2 = f_1\, dz_1$ on $U_1 \cap U_2 \Rightarrow f_2 = -z_1^2 f_1$ on $\mathbb{C} - \{0\}$

Show impossible if f_1 holom on $\mathbb{C}$, f_2 holom on $\mathbb{C} - \{0\} \cup \{\infty\}$