225B
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Homework II

1. A *Lie group* G is a group and a manifold such that the group operations of multiplication $G \times G \to G$ and inverse $G \to G$ are C^∞. A vector field on G is called *left invariant* if for each $g \in G$
$$V(g\, h) = dL_g(V(h))$$
where $L_g : G \to G$ is defined by $h \to gh$, $\forall h \in$
   (a) Show that for each vector $v \in T_eG$, e = identity of G, there is a unique left-invariant vector field V with $V(e) = v$.
   (b) Prove that all the integral curves of V are defined on $(-\infty, +\infty)$ (even if G is not compact) [Suggestion: Left multiplication takes an integral curve to an integral curve]
   (c) Show that the integral curve of V through e is a "one parameter subgroup" of G, i.e., $\gamma_e(t) \cdot \gamma_e(s) = \gamma_e(t+s)$ where $\cdot$ = G-multiplication, γ_e = integral curve at e, tangent = V.

2. Suppose $V(x,y,z) = P(x,y,z)\frac{\partial}{\partial x} + Q(x,y,z)\frac{\partial}{\partial y} + R(x,y,z)\frac{\partial}{\partial z}$ is a nowhere-zero C^∞ vector field on $\mathbb{R}^3$. Find the condition for the existence (locally) of a C^∞ function $\lambda(x,y,z)$ nowhere vanishing and such that
$$\lambda V = \text{gradient } F$$
for some C^∞ function F.

Suggestion: The gradient thing is going to happen if and only if $(\lambda V)^{\perp}$ is integrable (where $^{\perp}$ = orthogonal complement) because the level surfaces of F (if F exists) are integral submanifolds of $(\lambda V)^{\perp}$. Now $(\lambda V)^{\perp} = V^{\perp}$. So we are looking for $V^{\perp}$ to be integrable. The vectors $(-Q, P, 0)$ and $(0, -R, Q)$ are a basis for $V^{\perp}$ (supposing $Q \neq 0$). So compute Lie brackets, etc.

3. Consider problem 2 from the "classical" viewpoint by showing that

(a) $\operatorname{curl}(\lambda V) = \lambda \operatorname{curl} V + \nabla\lambda \times V$

~~and~~ so that

$\operatorname{curl}(\lambda V) = 0 \iff -\lambda \operatorname{curl} V = \nabla\lambda \times V$

and then a vector A can be written as $B \times C$ if and only if $A \perp C$. [You need to do the necessary condition here only for problem 2: proof of sufficiency is not asked for).

Hwk II (continued)

4 Let $SO(3)$ = the set of 3×3 orthogonal matrices with determinant $= +1$.

(a) Show that every element in $SO(3)$ is a rotation around some axis.

(b) Deduce that every element of $SO(3)$ belongs to a one-parameter subgroup.

(c) Show that this one-parameter subgroup has the $t \to e^{Bt}$ where B is a 3×3 skew-symmetric matrix and matrix exponentiation is given by

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{3!}M^3 + \cdots$$

(d) Prove that in general (any square matrix M, $n\times n$, the series for e^M converges

[Suggestion: Use "operator norm",

$$\| A \| = \sup\{ \| A\vec{x} \| : \|\vec{x}\| \le 1\}$$

and note that $\|A_1 A_2\| \le \|A_1\| \|A_2\|$)

5. (continuation of problem 4 and extension of it)

(a) Show that $t \to e^{tM}$ is a one-parameter subgroup of the group of $n\times n$ nonsingular matrices for any $n\times n$ matrix M (group $= Gl(n,\mathbb{R})$)

(b) Show that $t \to e^{tM}$ has all its elements in $SO(n)$ (orthogonal $n\times n$ matrices of determinant $= 1$) if and only if M is skew symmetric.

(c) Show that every element of $Gl(n,\mathbb{R})$ has the form e^M if the element is sufficiently close to the identity (i.e., $\exists$ a neighborhood U

of the identity such that every matrix in U has this form for some M).

[Suggestion: Use the series
$$\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots$$]

6. Show that a 1-parameter subgroup of $Gl(n,\mathbb{R})$ (which you may assume is C^∞) $t \to \varphi(t)$ (with $\varphi(t)\cdot\varphi(s) = \varphi(t+s)$ etc.) is uniquely determined by its tangent vector at e namely $V = \left.\frac{d\varphi(t)}{dt}\right|_{t=0}$

[Suggestion: Show that φ must be an integral of the left invariant vector field determined by V].

7. Deduce that every 1-parameter subgroup of $Gl(n,\mathbb{R})$ has the form $t \to e^{tM}$ and that every 1-parameter subgroup of $SO(n)$ has the form $t \to e^{tA}$, where A is skew-symmetric.

8 Work out details of proof that $\frac{d}{dt}\left(d\varphi_{-t}(W(\varphi_t(p))) - W(p)\right) = [V,W]_p$ where φ_t = flow of vector field V.

9. Write out Taylor expansion of integral curves in local coordinates to show (in general — done in class in specific cases)

W curve differs from W curve [V curve by $t^2[V,W]$ + h.o.

V curve