Notes for Feb. 2, 2009 — 2 parts: I Function linearity et al and II d, classical vector calculus, $d^2=0$, deRham cohomology

I Crucial Observation and Technique: Function linearity etc.

Consider the function on vector fields

$$I(X,Y) = X\,\omega(Y) - Y\,\omega(X) - \omega([X,Y])$$

where ω is a fixed C^∞ 1-form.

This is clearly "local" in the sense that $I(X,Y)$ at a point p is determined by X and Y is any neighborhood of p. But at first site, it does not appear to be "pointwise" or "tensorial" in X and Y in the sense that $I(X,Y)$ at p is determined by $X|_p$ and $Y|_p$.

The basic observation is this: Suppose I additively linear ($I(X_1+X_2,Y) = I(X_1,Y)+I(X_2,Y)$) (similarly for Y) and local as defined above. Then

$I(X,Y)|_p$ depends only on $X|_p$ and $Y|_p$

$\Longleftrightarrow$ I is function-linear (at p) in the sense that

$$I(fX,Y)|_p = f(p)\, I(X,Y)|_p$$

and

$$I(X,gY)|_p = g(p)\, I(X,Y)|_p,$$

for all C^∞ f,g defined in a neighborhood of p.

Proof of observation: Write $X=\sum f_i \frac{\partial}{\partial x_i}$, $Y=\sum g_j \frac{\partial}{\partial x_j}$. Function linearity gives (along with additive linearity)

$$I(X,Y)|_p = \sum f_i(p)\, g_j(p)\, I\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\Big|_p$$

Thus $I(X,Y)|_p$ is determined by $X|_p$, $Y|_p$.
(Other direction of implication is clear).

How to check function linearity of our particular I (in X)

$$I(fX,Y) = (fX)\omega(Y) - Y\omega(fX) - \omega([fX,Y])$$

$$= (fX)\omega(Y) - Y(f\omega(X)) - \omega(f[X,Y] - Yf(X))$$

$$= f(X\omega(Y) - Y\omega(X)) - (Yf)\omega(X) - f\omega([X,Y]) + (Yf)\omega(X)$$

$$= f\, I(X,Y)$$

Note that the "bad terms" $\pm(Yf)\omega(X)$, which involve derivatives of f cancel out!

Now function linearity means that we can check the formula

$$d\omega(X,Y) = I(X,Y)$$

(where d is $d_{(x_1,\dots x_n)}$ in loc. coords)

by checking $X = \partial/\partial x_\ell$ $Y = \partial/\partial x_k$ only. For this case, assuming wolog $\ell < k$:

$d_{(x_1..x_n)}(\Sigma F_i dx_i)$ applied to $\partial/\partial x_\ell, \partial/\partial x_k$

$$\overset{\text{def}}{=} \left(\sum_{i<j}\left(-\frac{\partial F_i}{\partial x_j} + \frac{\partial F_j}{\partial x_i}\right) dx_i \wedge dx_j\right)\left(\frac{\partial}{\partial x_\ell}, \frac{\partial}{\partial x_k}\right)$$

$$= -\frac{\partial F_\ell}{\partial x_k} + \frac{\partial F_k}{\partial x_\ell}$$

$$= \underset{\partial/\partial x_\ell}{\underset{\shortparallel}{X}}\ \underset{F_k}{\underset{\shortparallel}{\omega(Y)}} - \underset{\frac{\partial}{\partial x_k}}{\underset{\shortparallel}{Y}}\ \underset{F_\ell}{\underset{\shortparallel}{\omega(X)}} - \overset{0}{\overset{\shortparallel}{\omega([X,Y])}}$$

So formula holds in this case, hence always $\square$

Corollary: $d_{(x_1\dots x_n)}$ is independent of choice of local coordinates.

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I Relationship of d to classical vector calculus and related matters ($d^2=0$ etc.)

Associate (on $\mathbb{R}^3$)

$$P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z} \overset{①}{\longleftrightarrow} P\,dx+Q\,dy+R\,dz$$

and

$$P\frac{\partial}{\partial x}+Q\frac{\partial}{\partial y}+R\frac{\partial}{\partial z} \overset{②}{\longleftrightarrow} P\,dy\wedge dz - Q\,dx\wedge dz + R\,dx\wedge dy$$

and

$$f(x,y,z)\,dx\wedge dy\wedge dz \overset{③}{\longleftrightarrow} f(x,y,z)$$

Then

grad f = vector field associated via ① to df

curl V = vector field associated via ② to d(1-form associated to V via ①)

(V a vector field)

and div V = function associated via ③ with 3-form obtained by d(2 form associated to V via ②).

(V a vector field)

Proof: Compute.

The idea that $d^2=0$ (which we shall check in general momentarily) corresponds to two items from vector calculus: curl(grad( ))=0 and div(curl(·))=0.

(Exercise).

[Note that Laplacian = div(grad(·)) does not directly fit this since no (nonzero) second order operator arises directly from d since $d^2=0$.

But one can get Laplacian by doing the associations in a peculiar order: Given a function f, look not at the 1-form attached to grad f by ① but at the 2-form attached to grad f by ②. Then d of this two-form (2-form) is associated via ③ to Laplacian of f:

$$f \to \frac{\partial f}{\partial x}\, dy \wedge dz - \frac{\partial f}{\partial y}\, dx \wedge dz + \frac{\partial f}{\partial z}\, dx \wedge dy$$

and $d(\ ) = \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dx \wedge dy \wedge dz$

$= \text{Laplacian}(f)\ dx \wedge dy \wedge dz.$

As we shall see later, this has a generalization to manifolds, but it involves Riemannian metric choice.]

Reason that $d \circ d$ = 0] Really Important Fact!!

d: k forms to k+1-forms

d: k+1 forms to k+2 forms

on any manifold, all k.

Compute in local coordinates: $\omega = \sum_{i_1 < i_2 < \cdots < i_k} f_{i_1 \ldots i_k}\, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$

Then
$$d\omega = \sum_{i_1 < \cdots < i_k,\, j} \frac{\partial f_{i_1 \ldots i_k}}{\partial x_j}\, dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

So
$$d(d\omega) = \sum_{\substack{j, \ell \\ i_1 < \cdots < i_k}} \frac{\partial^2 f_{i_1 \ldots i_k}}{\partial x_j \partial x_\ell}\, dx_\ell \wedge dx_j \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

$= 0$ since $\sum_{j,\ell} \frac{\partial^2 f_{i_1 \ldots i_k}}{\partial x_j \partial x_\ell}\, dx_\ell \wedge dx_j = 0$ ($i_1, i_2 \cdots i_k$ fixed)

Since "$d^2 = 0$", $\quad \operatorname{im} d_{k-1 \text{ to } k} \subset \ker d_{k \text{ to } k+1}$

This makes it natural to look at

$$\frac{\ker d_{k \text{ to } k+1}}{\operatorname{im} d_{k-1 \text{ to } k}} \overset{\text{def.}}{=} \text{deRham } k\text{-cohomology} \overset{\text{notation}}{=} H^k(M, \mathbb{R})$$

Example: $\mathbb{R}^2 - \{(0,0)\}$, $k = 1$.

Given 1-form ω with $d\omega = 0$, $\omega = df$ for some function f if and only if

$$\oint_{\text{unit circle counterclockwise}} \omega = 0$$

So (since $\exists$ 1-form with $\oint \neq 0$, namely "$d\theta$")

$$H^1(\mathbb{R}^2 - \{(0,0)\}) \cong \mathbb{R}.$$

Exercise: Prove the "if and only if" $\oint = 0$