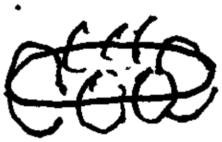
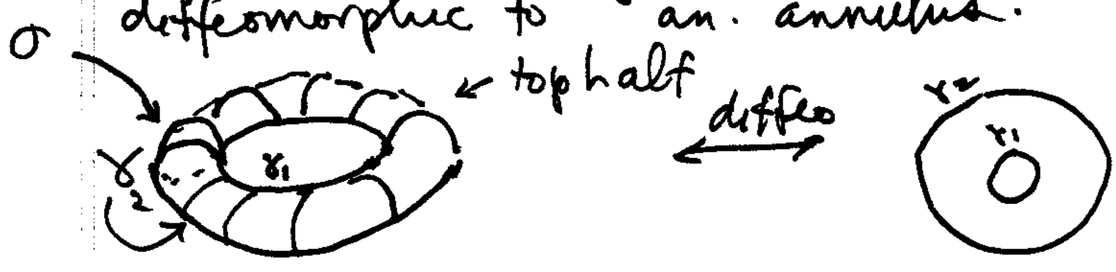
Calculation of deRham 1-cohomology of
compact, orientable surfaces:
$H^1_{deRham}(\text{genus } g \text{ surface}, \mathbb{R}) \cong \mathbb{R}^{2g}$

S^2: union of $U = S^2 - \{(0,0,-1)\}$ and $V = S^2 - \{(0,0,1)\}$.
U, V open sets diffeomorphic to $\mathbb{R}^2$
(explicit diffeomorphism: "stereographic projection"
 for U,  for V).

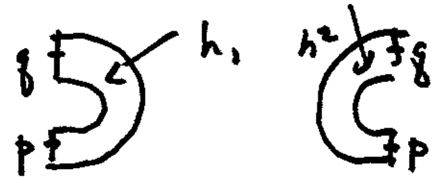
So given ω a 1-form with $d\omega = 0$,
$\exists\ f_1 : U \to \mathbb{R} \ni \quad df_1 = \omega$ on U
$\exists\ f_2 : V \to \mathbb{R} \ni \quad df_2 = \omega$ on V.
Changing f_2 to $f_2 + c$ for suitable $c \in \mathbb{R}$ we
can assume $f_1((1,0,0)) = f_2((1,0,0))$. Then
since $U \cap V$ is (arcwise) connected, $f_1 \equiv f_2$ on
$U \cap V$. So $F = f_1 \cup f_2$ (in obvious sense)
is C^∞ on S^2 and has $dF = \omega$. So
$H^1_{deRham}(S^2, \mathbb{R}) = 0$.

Torus:  Slicing torus of revolution (around
z axis rotate ) by

horizontal plane gives two pieces each
diffeomorphic to an annulus.



Given ω 1-form with $d\omega = 0$, $\exists f_1$ on top half with $df_1 = \omega$ if and only if $\oint_{\gamma_1} \omega = 0$ where γ_1 = central circle as shown

(calculus result: do 

make $h_1 = h_2$ at , $h_1 \cup h_2$ is defined and C^∞ $\iff h_1(q) = h_2(q) \iff \oint_{\gamma_1} \omega = 0$).

Supposing $\oint_{\gamma_1} \omega = 0$, have f_1 on top half and f_2 on bottom half, wolog $f_1 = f_2$ at one point of γ, hence on all of γ. Then f_1 and f_2 agree on outside circle γ_2 if and only if integrating ω "over the top" = integrating ω "under the bottom" $\iff$ $\oint_\sigma \omega = 0$

$\sigma =$ 

Combining these observations we see that

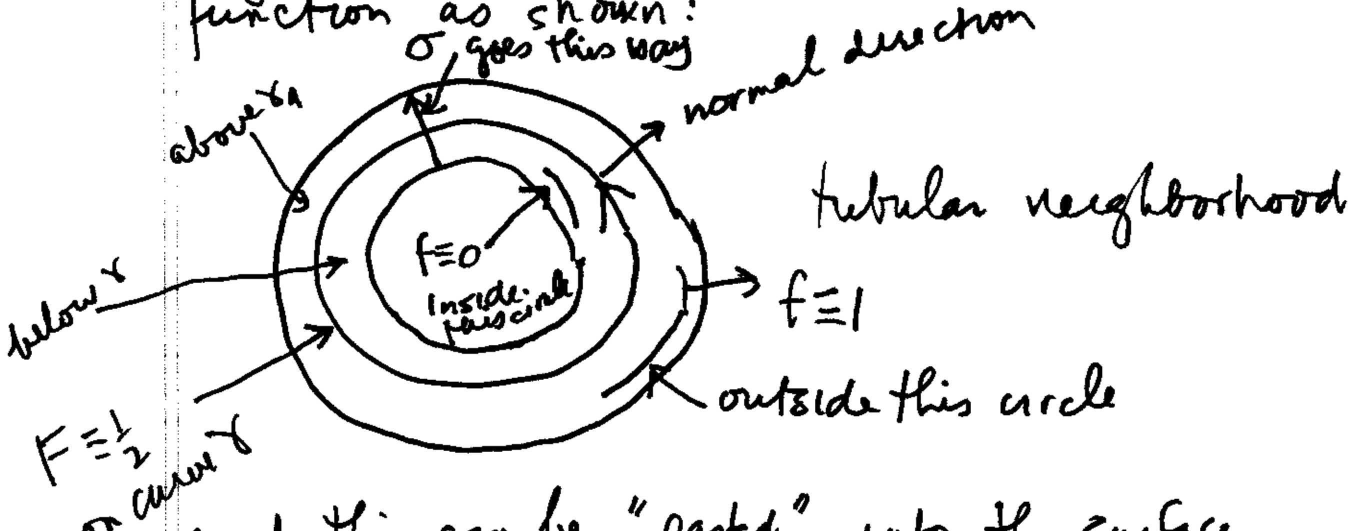
closed forms $\xrightarrow{\quad} \mathbb{R}^2$ defined by (P for "periods")

$\omega \xrightarrow{P} \left(\oint_{\gamma_1} \omega, \oint_\sigma \omega\right)$ has kernel

exactly = exact forms so $P: \dfrac{\text{closed 1-forms}}{\text{exact 1-forms}} \to \mathbb{R}^2$ is injective.

So to show that P is an isomorphism, we need only show that P is surjective.

For this, note that if γ is a smoothly closed simple curve (with an orientation) in an oriented surface then we can define a closed form in a neighborhood of γ by taking dF where F is a function as shown:

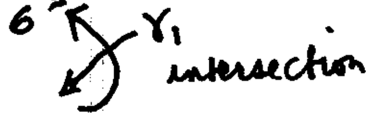


and this can be "pasted" into the surface (since $dF \equiv 0$ outside the shrunken tubular neighborhood). We do this for γ_1: call the resulting form ω_1. And for σ: call the form ω_2.

Note that $\oint_\sigma \omega_1 = \pm 1$ (actually $+1$ as we have done ω_1)

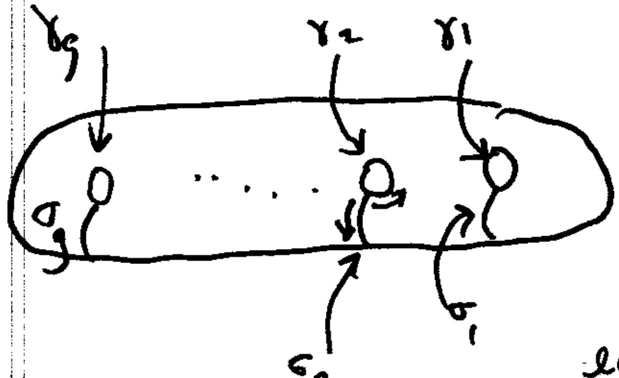
and $\oint_{\gamma_1} \omega_1 = 0$

while $\oint_\sigma \omega_2 = 0$ and $\oint_{\gamma_1} \omega_2 = \pm 1$



So image of P contains $(0,1)$ and $(1,0)$ so image $P = \mathbb{R}^2$.

It is not difficult to extend this argument to arbitrary (finite) genus g. A g-hole surface can be built by gluing two copies of the planar region shown together along the outside curve and the "inside" curves $\gamma_1 \ldots \gamma_g$ each to the corresponding γ on the other copy.



On each planar region separately we can find, given a closed form ω, a function f_{top} with $df_{top} = \omega$ if and only if

$$\oint_{\gamma_j} \omega = 0 \qquad j = 1, 2 \ldots, g$$

(Easy argument

 patches with 

if and only if $\oint_{\gamma_1} \omega = 0$ etc.)

Then f_{top} and f_{bottom} (which also exists if and only if $\oint_{\gamma_j} \omega = 0$, $j = 1 \ldots g$) can be taken to along the outside curve (replacing f_{bottom} by $f_{bottom} + c$, $c \in \mathbb{R}$ if need be). They then agree, $f_{top} = f_{bottom}$ on each γ_j, $j = 1 \ldots g$, if and only if

$$\oint_{\sigma_j} \omega = 0$$

So as before we get an injective map

$$\frac{\text{closed 1-forms}}{\text{exact 1-forms}} \xrightarrow{P} \mathbb{R}^{2g}$$

by sending $[\omega] \to \left(\oint_{\gamma_1}\omega, \dots, \oint_{\gamma_g}\omega, \oint_{\sigma_1}\omega, \dots, \oint_{\sigma_g}\omega\right)$

To see that this is onto, we construct $\omega_1, \dots, \omega_g$ supported in tubular neighborhood of $\gamma_1, \dots, \gamma_g$ respectively with

$\oint_{\gamma_j}\omega_j = 0$ and $\oint_{\sigma_j}\omega_j = \pm 1$ while

$\oint_{\gamma_j}\omega_k = 0$ and $\oint_{\sigma_k}\omega_k = 0$ $k \neq j$ (because ω_k is supported in small nbhd of γ_k)

Similarly we get $\theta_1, \dots, \theta_g$ supported near $\sigma_1, \dots, \sigma_g$ with corresponding properties

$\left(\oint_{\gamma_j}\theta_j = \pm 1, \oint_{\sigma_j}\theta_j = 0, \oint_{\gamma_j}\theta_k = 0 = \oint_{\sigma_j}\theta_k \text{ if } k \neq j\right)$.

Then $[\omega_1], \dots [\omega_g], [\sigma_1], \dots [\sigma_g]$ have linearly independent P-images in $\mathbb{R}^{2g}$.
So P is injective and surjective and

$$P : H^1_{\text{deRham}}(\text{genus } g \text{ surface}) \to \mathbb{R}^{2g}$$