Coordinate Invariance of d on 1-forms

We define $d\omega$, $\omega = \sum f_i dx_i$ in local coordinates $(x_1, \ldots, x_n)$ by

$$d\omega = \sum_{i<j} \left(-\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i} \right) dx_i \wedge dx_j$$

(this is motivated by $d(f\,dx_i)$ should be $df \wedge dx_i$). For emphasis, let us call this $d_{(x_1,\ldots,x_n)}\omega$ since in principle it depends on the choice of local coordinates $(x_1, \ldots x_n)$. We want to show that $d_{(x_1,\ldots x_n)}\omega = d_{(\hat{x}_1,\ldots \hat{x}_n)}\omega$ in the sense that for the right-hand side we write ω in $(\hat{x}_1, \ldots \hat{x}_n)$ coordinates and compute d with the same formula as in the third line above - except with $\hat{}$ coordinates. Now

$$dx_i = \sum_l \frac{\partial x_i}{\partial \hat{x}_l} d\hat{x}_l \quad \text{so} \quad \omega = \sum_{i,l} f_i \frac{\partial x_i}{\partial \hat{x}_l} d\hat{x}_l$$

$$= \sum_i \left(\sum_l f_l \frac{\partial x_l}{\partial \hat{x}_i} \right) d\hat{x}_i$$

So $\omega = \sum \hat{f}_i d\hat{x}_i$ where $\hat{f}_i = \sum_l f_l \frac{\partial x_l}{\partial \hat{x}_i}$

Thus

$$d_{(\hat{x}_1,\ldots\hat{x}_n)}\omega = \sum_{i<j} \left(-\frac{\partial \hat{f}_i}{\partial \hat{x}_j} + \frac{\partial \hat{f}_j}{\partial \hat{x}_i} \right) d\hat{x}_i \wedge d\hat{x}_j$$

and

$$\frac{\partial \hat{f}_i}{\partial \hat{x}_j} = \frac{\partial}{\partial \hat{x}_j} \left(\sum_l f_l \frac{\partial x_l}{\partial \hat{x}_i} \right) = \sum_l \left(\frac{\partial f_l}{\partial \hat{x}_j} \frac{\partial x_l}{\partial \hat{x}_i} + f_l \frac{\partial^2 x_l}{\partial \hat{x}_j \partial \hat{x}_i} \right)$$

while $$\frac{\partial \hat{f}_j}{\partial \hat{x}_i} = \sum_\ell \frac{\partial f_\ell}{\partial \hat{x}_i}\frac{\partial x_\ell}{\partial \hat{x}_j} + f_\ell \frac{\partial^2 f_\ell}{\partial \hat{x}_i \partial \hat{x}_j}$$ (by ② interchanging i,j in previous)

so $$\frac{\partial \hat{f}_j}{\partial \hat{x}_i} - \frac{\partial \hat{f}_i}{\partial \hat{x}_j} = \sum_\ell - \frac{\partial f_\ell}{\partial \hat{x}_j}\frac{\partial x_\ell}{\partial \hat{x}_i} + \frac{\partial f_\ell}{\partial \hat{x}_i}\frac{\partial x_\ell}{\partial \hat{x}_j}) + 0$$

2nd derivative terms cancel

Hence $$d\omega_{(\hat{x}_1 \dots \hat{x}_n)} = \sum_{i<j}\left(\sum_\ell - \frac{\partial f_\ell}{\partial \hat{x}_j}\cdot\frac{\partial x_\ell}{\partial \hat{x}_i} + \frac{\partial f_\ell}{\partial \hat{x}_i}\frac{\partial x_\ell}{\partial \hat{x}_j}\right) d\hat{x}_i \wedge d\hat{x}_j$$

$$= \sum_{i<j}\left(\sum_{k,\ell} - \frac{\partial f_\ell}{\partial x_k}\frac{\partial x_k}{\partial \hat{x}_j}\frac{\partial x_\ell}{\partial \hat{x}_i} + \frac{\partial f_\ell}{\partial x_k}\cdot\frac{\partial x_k}{\partial \hat{x}_i}\frac{\partial x_\ell}{\partial \hat{x}_j}\right) d\hat{x}_i \wedge d\hat{x}_j$$

$$= \frac{1}{2}\sum_{\substack{\text{any}\\ i,j}}\left[\sum_{\ell,k}\left(- \frac{\partial f_\ell}{\partial x_k}\frac{\partial x_k}{\partial \hat{x}_j}\frac{\partial x_\ell}{\partial \hat{x}_i} + \frac{\partial f_\ell}{\partial x_k}\frac{\partial x_k}{\partial \hat{x}_i}\frac{\partial x_\ell}{\partial \hat{x}_j}\right) d\hat{x}_i \wedge d\hat{x}_j\right.$$

$= dx_\ell$ when i is summed out

$$= \frac{1}{2}\sum_{\ell,k} - \frac{\partial f_\ell}{\partial x_k}\left(\sum_{i,j}\left(\frac{\partial x_\ell}{\partial \hat{x}_i} d\hat{x}_i \wedge \frac{\partial x_k}{\partial \hat{x}_j} d\hat{x}_j\right)\right)$$ dx_k when j is summed out

$$+ \frac{1}{2}\sum_{\ell,k}\frac{\partial f_\ell}{\partial x_k}\left(\sum_{i,j}\frac{\partial x_k}{\partial \hat{x}_i} d\hat{x}_i \wedge \frac{\partial x_\ell}{\partial \hat{x}_j} d\hat{x}_j\right))$$

dx_k when i summed; dx_ℓ when j summed

$$= \frac{1}{2}\sum_{\ell,k}\left(- \frac{\partial f_\ell}{\partial x_k} dx_\ell \wedge dx_k + \frac{\partial f_k}{\partial x_\ell} dx_k \wedge dx_\ell\right.$$

$$= \sum_{k<\ell}\left(- \frac{\partial f_\ell}{\partial x_k} + \frac{\partial f_k}{\partial x_\ell}\right)(dx_k \wedge dx_\ell) = d_{(x_1 \dots x_n)}\omega .$$

This is a bit messy, but it does work.

The messiness justifies the process of isolating (coord independent) characterizing properties of d and then verifying that d in local coordinates has the characterizing properties and hence is coordinate independent.

Alternatively, one can show that at $p \in M$, $\omega = \sum f_i dx_i$

$$d\omega_{(x_1..x_n)}(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])$$

X, Y vector fields in nbhd of p

$(x_1, .. x_n)$ coordinates around p

Then since RHS is coordinate independent, so is left hand side. To check this formula, note:

(1) It holds if $X = \partial/\partial x_i$, $Y = \partial/\partial x_j$, $i<j$: left hand side $= -\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}$ while $\omega(Y) = f_j$ $\omega(X) = f_i$

so $X\omega(Y) - Y\omega(X) - \omega(\overset{=0}{[X,Y]}) = \frac{\partial f_j}{\partial x_i} - \frac{\partial f_i}{\partial x_j}$ ✓

(2) $RHS(fX,Y) = f\, RHS(X,Y)$ and similarly $RHS(X,fY) = f\, RHS(X,Y)$.

proof: $(fX)\omega(Y) = f\,(X\omega(Y))$

$$-Y\omega(fX) = -Y(f\omega(X)) = -Yf\,\omega(X) - fY\omega(X)$$

$$-\omega([fX,Y]) = -\omega((-Yf)X + f[X,Y]) = +(Yf)\,\omega(X) - f\omega[X,Y].$$

So Yf terms cancel when RHS is added up, f factors out as required.

Of course $f(\mathrm{LHS})(fX,Y) = f\,\mathrm{LHS}(X,Y)$ and similarly for X, fY.
Points (1) and (2) (and additive linearity in X and Y of both sides) thus imply that $\mathrm{LHS} = \mathrm{RHS}$ since both give same answers on $X = \sum f_i \frac{\partial}{\partial x_i}$ and $Y = \sum g_j \frac{\partial}{\partial x_j}$, namely

$$\text{both sides} = \sum_{i,j} f_i g_j \ \underset{\text{RHS}}{\overset{\text{LHS}}{\text{or}}} \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$$

(which at p depends only on $f_i(p)$ and $g_j(p)$).

This may seem a bit obscure at first! Think it over. But it does show that $d_{(x_1,\dots x_n)}\omega$ at p is the same as $d\omega$ calculated at p in any other local coordinate system.

This is all a lot of trouble but d is really important so the trouble is worth going through! It also has a reasonably straightforward generalization to k-forms, $k>2$: $d(f\,dx_{i_1}\wedge\dots\wedge dx_{i_k}) = df\wedge dx_{i_1}\wedge\dots\wedge dx_{i_k}$ as definition and

$$d\omega(X_0, X_1, \dots X_k) = \sum (-1)^i X_i\, \omega(X_0 \dots \underset{\text{omitted}}{\hat{X_i}} \dots X_k) + \sum (-1)^{i+j} \omega([X_i, X_j], X_0 \dots \underset{i\ \&\ j\ \text{omitted}}{\dots})$$

It is worthwhile to work out some specific instances. For example, suppose $\omega = \frac{1}{2} r^2 d\theta$ in polar coordinates on $\mathbb{R}^2 - \{(0,0)\}$. Then $d_{(r,\theta)}\omega = r\, dr \wedge d\theta$.

In (x,y) coordinates, $dr = \frac{\partial r}{\partial x} dx + \frac{\partial r}{\partial y} dy = \frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy$

and $d\theta = \frac{\partial \theta}{\partial x} dx + \frac{\partial \theta}{\partial y} dy = -\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy$.

So $\omega = \frac{1}{2}\left((x^2+y^2)^{\frac{1}{2}}\right)^2 \left(-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy\right) = -\frac{y}{2} \cancel{dx} + \frac{x}{2} dy$

So $d_{(x,y)}\omega = -\frac{1}{2} dy \wedge dx + \frac{1}{2} dx \wedge dy = dx \wedge dy$

while $r\, dr \wedge d\theta = \sqrt{x^2+y^2}\left(\frac{x}{\sqrt{x^2+y^2}} dx + \frac{y}{\sqrt{x^2+y^2}} dy\right) \wedge \left(\frac{-y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy\right)$

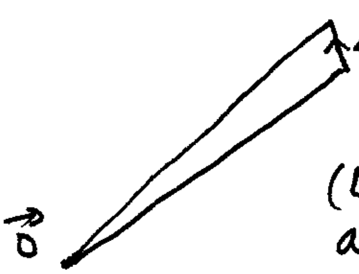
$= \sqrt{x^2+y^2}\left(\frac{x^2}{(x^2+y^2)^{3/2}} dx \wedge dy - \frac{y^2}{(x^2+y^2)^{3/2}} dy \wedge dx\right)$

$= \frac{(x^2+y^2)\sqrt{x^2+y^2}}{(x^2+y^2)^{3/2}} dx \wedge dy = dx \wedge dy$.

This confirms the coordinate invariance of d in this instance.

The geometric interpretation is of interest: By Green's/Stokes Theorem, $\oint_\gamma \omega$ around a simple closed curve $\gamma = \int_{\text{interior of } \gamma} dx \wedge dy$

if γ is oriented counterclockwise



$(\Delta x, \Delta y)$ in (x,y) coordinates

$(\Delta r, r\Delta\theta)$ in (r,θ) coordinates

area of triangle $= \Delta A = (x,y) \times (\Delta x, \Delta y)$

$= x\Delta y - y \Delta x$

also area of triangle $= \frac{1}{2} r \cdot (r\Delta\theta)$ ← part $\perp$ radius, Δr does not count

$= \frac{1}{2} r^2 \Delta\theta$

So $\omega(\gamma'(t)) = \frac{dA}{dt}$ in both coordinate systems, if A = area swept out.