Basic Problematical Example for Frobenius Theorem

$$X = \frac{\partial}{\partial x} \qquad Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} \quad \text{on } \mathbb{R}^3$$

If S were an integral submanifold for span (X, Y) passing through $(0,0,0)$, then the curve $\gamma(t) = (t, 0, 0) \subset S$. Then so is $\Theta_t(s) = (t, s, ts) \subset S$. Moreover $\psi(s) = (0, s, 0) \subset S$ and hence so is $\eta_s(t) = (t, s, 0)$. Now fix $t > 0$ and consider the curve $\Gamma(s) = (t, s, ts) \subset S$. When $s = 0$, this curve has tangent $(0, 1, t)$ so $(0, 1, t) \in T_{(t,0,0)}S$.

But $s \to \eta_s(t)$ has (as a function of s) tangent vector $(0, 1, 0)$ so $(0, 1, 0) \in T_{(t,0,0)}S$.

Thus $(0, 0, t) \in T_{(t,0,0)}S$ since $T_{(t,0,0)}S$ is a subspace. But $(0, 0, t) \notin \text{span}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} + t\frac{\partial}{\partial z}\right)$ (when $t > 0$). This is a contradiction of the supposed existence of S.