Applications of deRham Cohomology

The results that follow will be stated for smooth maps. But many of them immediately imply similar results for continuous maps via two basic principles: (a) Every continuous map $F: M_1 \to M_2$ from one differentiable manifold to another is homotopic to a smooth map and (b) two smooth maps $F_1: M_1 \to M_2$ and $F_2: M_1 \to M_2$ that are continuously homotopic (homotopic as continuous maps) are smoothly homotopic, i.e. via a homotopy $H: M_1 \times [0,1] \to M_2$ that is smooth in the sense of being the restriction to $M_1 \times [0,1]$ of some smooth map $M_1 \times (-\varepsilon, 1+\varepsilon) \to M_2$, some $\varepsilon > 0$.

For the applications that follow, we make frequent use of the following: If $F_1: M_1 \to M_2$ and $F_2: M_1 \to M_2$ are smooth maps that are smoothly homotopic, then $F_1^*: H^k_{deR}(M_2, \mathbb{R}) \to H^k_{deR}(M_1, \mathbb{R})$ and $F_2^*: H^k_{deR}(M_2, \mathbb{R}) \to H^k_{deR}(M_1, \mathbb{R})$ are equal. This fact follows by exactly the same argument (using $\mathcal{L}_X = di_X + i_X d$ etc.) that we used to prove the Poincare / Volterra Lemma.

Theorem: The identity map $I: S^n \to S^n$ is not smoothly homotopic to a constant map $C: S^n \to S^n$; $C(S^n) =$ a one-point set ($n \geq 1$).

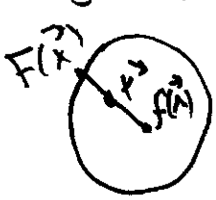
Proof: $C^*: H^n_{deR}(S^n, \mathbb{R}) \to H^n_{deR}(S^n, \mathbb{R})$ is the 0-map since $C^*(\omega) = 0$ for all n forms ω on S^n (calculation). But Id^* is the identity and $H^n_{deR}(S^n, \mathbb{R}) \neq \{0\}$ ($H = \mathbb{R}$). Hence Id is not smoothly homotopic to C. □

Corollary ("No-Retraction Theorem"): There is no smooth map $F: B^{n+1} \to S^n$ with $F|S^n =$ identity on S^n. (Here $B^{n+1} = \{\vec{x} \in \mathbb{R}^{n+1} : \|\vec{x}\| \leq 1\}$).

Proof. If such F existed then $H(\vec{x}, t) \stackrel{\text{def}}{=} F(t\vec{x})$ $\vec{x} \in S^n$ would be a homotopy of Id on S^n to the constant map of S^n to $F(\vec{0})$. □

Corollary (Brouwer Fixed Point Theorem): Every smooth map $f: B^{n+1} \to B^{n+1}$ ($n \geq 0$) has a fixed point.

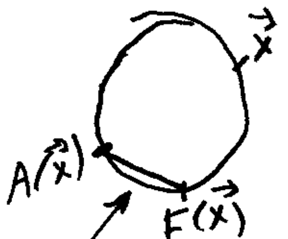
Proof: $n=0$ case follows from Intermediate Value Theorem. If $f: B^{n+1} \to B^{n+1}$ ($n \geq 1$) has fixed point, define $F: B^{n+1} \to S^n$ by $F(\vec{x}) =$ point of intersection with S^n of directed line segment from $f(\vec{x})$ to $\vec{x}$. This is a "retraction" onto S^n, as prohibited by "no-retraction" theorem. □



The $_\wedge^{\text{next}}$ topic is maps of spheres to themselves:

Theorem 1: Every smooth map $F: S^{2n} \to S^{2n}$ that is homotopic to the identity has a fixed point.

Proof: If F has no fixed point, then F is $_\wedge^{\text{(smoothly)}}$ homotopic to the antipodal map $A(\vec{x}) = -\vec{x}$. This follows by "projecting" on to S^{2n} the straight line homotopy in $\mathbb{R}^3$ from F to A. Since $F(\vec{x}) \neq \vec{x}$, the straight line segment from $F(\vec{x})$ to $A(\vec{x})$ does not contain $\vec{0}$ so the projection



$\lambda A(\vec{x}) + (1-\lambda) F(\vec{x})$, $\lambda \in [0,1]$

$$\frac{\lambda A(\vec{x}) + (1-\lambda) F(\vec{x})}{\| \lambda A(\vec{x}) + (1-\lambda) F(\vec{x}) \|}$$

is defined and smooth. Now $A^*: H^{2n}_{deR}(S^{2n}, \mathbb{R}) \to H^{2n}_{deR}(S^{2n}, \mathbb{R})$ is the map from $\mathbb{R}$ to $\mathbb{R}$ consisting of multiplication by -1. So if F is homotopic to A, then F^* is also multiplication by -1. On the other hand, F homotopic to the identity implies F^* is the identity map from $\mathbb{R}$ to $\mathbb{R}$. Hence F is not homotopic to A if F is homotopic to the identity, and F must have a fixed point. $\square$.

since [volume form from $\mathbb{R}^{2n+1}$] generates H^{2n} & A^* vol form $= -$ vol form ...

Corollary (already known from linear algebra): Every element of $SO(2n+1)$ has an axis, that is, for every $A \in SO(2n+1)$, $\exists$ $v \in S^{2n}$ with $Av = v$.

Proof: $SO(2n+1)$ is connected so every element is connected to the identity by a smooth curve, which can be thought of as a homotopy from a given element to the identity. To see that $SO(N)$ is connected for all N (and hence $N = 2n+1$), reason inductively: Assuming $SO(N-1)$ is connected, $SO(N)$ must be connected because $\{A \in SO(N) : A(0,..0,1) = \vec{v}\}$, $\vec{v} \in S^{N-1}$, is homeomorphic to $SO(N-1)$ and $SO(N)$'s connectivity follows from that of $SO(N-1)$ and S^{N-1} and this observation (details as exercise).

Note that there exist maps $S^{2n+1} \to S^{2n+1}$, indeed elements of $SO(2n+2)$, which have no fixed points but are homologic to the identity: this is possible because the antipodal map $A : S^{2n+1} \to S^{2n+1}$ is homotopic to the identity. This is in turn possible because $A^* : H^{2n+1}_{deR}(S^{2n+1}, \mathbb{R}) \to H^{2n+1}_{deR}(S^{2n+1}, \mathbb{R})$ is the identity map.

Example: $(x_1, \ldots x_{2n+2}) \to (-x_2, x_1, -x_4, x_3 \ldots, -x_{2n+2}, x_{2n+1})$ with the homotopy to the identity being obtained via rotation in the $x_1\, x_2$ coordinates, in the $x_3\, x_4$ coordinates, .., in x_{2n+1}, x_{2n+2} coordinates.

Corollary: Every (tangent) vector field on S^{2n} has a zero.

Proof: If X is a vector field without zeroes, then φ_t, t small positive, φ_t the flow of X, is homotopic to the identity but has no fixed points.

It is worth noting that the applications we have made here do not actually require very much detailed information about $H^*_{deRham}(S^n, \mathbb{R})$ — we used more than we actually needed: For the results on p. 2, only $H^n_{deR}(S^n, \mathbb{R}) \neq 0$ was really needed. For the fixed point and vector field results, only $A^*: H^{2n}_{deR}(S^{2n}, \mathbb{R}) \to H^{2n}_{deR}(S^{2n}, \mathbb{R})$ is not the identity map was needed. But both of these follow easily just from Stokes' Theorem: our inductive proof that $H^k_{deR}(S^k, \mathbb{R}) \cong \mathbb{R}$, and $H^k_{deR}(S^k, \mathbb{R})$ is generated by [vol. form of S^k] were not really required. Namely, let ω_0 = the volume form on $S^k \subset \mathbb{R}^{k+1}$ induced from the $\mathbb{R}^{k+1}$ volume form Ω by $\omega_0\big|_{\vec{x}} = \Omega(\vec{x}, \underbrace{\cdots}_{k \text{ slots}})$. Then $\int_{S^k} \omega_0 > 0$.

So $[\omega_0] \neq 0$ since if $\omega_0 = d\theta$ then $\int_{S^k} \omega_0 = \int_{S^k} d\theta = 0$.

So $H^k_{deR}(S^k, \mathbb{R}) \neq 0$. Moreover $A^*[\omega_0] = [A^*\omega_0]$ and $A^*\omega_0 = (-1)^{k+1}\omega_0$. So if k is even, A^* on deRham classes $\neq$ identity. A lot comes from a little (namely Stokes' Theorem) here. Of course, the big power is the homotopy invariance of F^* (from Poincaré Lemma proof)