Yet another example of how the Poincare Lemma proof construction works out: closed 2-form on $\mathbb{R}^3$ (in general)

$$\omega = P(x,y,z)\, dy \wedge dz - Q^{(x,y,z)} dx \wedge dz + R(x,y,z)\, dx \wedge dy$$

$$d\omega = 0 \iff \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = 0.$$

$$H((x,y,z),t) = (tx, ty, tz) \qquad \text{(contraction to } (0,0,0))$$

Then $H_*\left(\frac{\partial}{\partial x}\right) = t\frac{\partial}{\partial x}$ $\quad H_*\left(\frac{\partial}{\partial y}\right) = t\frac{\partial}{\partial y}$ $\quad H_*\left(\frac{\partial}{\partial z}\right) = t\frac{\partial}{\partial z}$

$$H_*\left(\frac{\partial}{\partial t}\right) = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} \quad \text{(as usual)}$$

We are interested in $i_{\frac{\partial}{\partial t}} H^*\omega$

To compute this, we find $\left(i_{\frac{\partial}{\partial t}} H^*\omega\right)\left(\frac{\partial}{\partial x}\right)$, $\left(i_{\frac{\partial}{\partial t}} H^*\omega\right)\left(\frac{\partial}{\partial y}\right)$, $\left(i_{\frac{\partial}{\partial t}} H^*\omega\right)\left(\frac{\partial}{\partial z}\right)$

$$\left(i_{\frac{\partial}{\partial t}} H^*\omega\right)\left(\frac{\partial}{\partial x}\right) = \omega\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}, \frac{\partial}{\partial x}\right) = zt\, Q\big|_{(tx,ty,tz)} - yt\, R\big|_{(tx,ty,tz)}$$

$$\left(i_{\frac{\partial}{\partial t}} H^*\omega\right)\left(\frac{\partial}{\partial y}\right) = -ztP + Rxt, \quad P, R \text{ evaluated at } (tx,ty,tz)$$

$$\left(i_{\frac{\partial}{\partial t}} H^*\omega\right)\left(\frac{\partial}{\partial z}\right) = ytP - xtQ, \quad P, Q \text{ evaluated at } (tx,ty,tz)$$

So

$$i_{\frac{\partial}{\partial t}} H^*\omega = t\Big(\left(zQ\big|_{(tx,ty,tz)} - yR\big|_{(tx,ty,tz)}\right)dx + \left(xR\big|_{(tx,ty,tz)} - zP\big|_{(tx,ty,tz)}\right)dy + \left(yP\big|_{(tx,ty,tz)} - xQ\big|_{(tx,ty,tz)}\right)dz\Big)$$

The "Poincare expectation" is that ($F_t(\cdot) = H(\cdot, t)$ as usual)

$$\omega = F_1^*\omega - F_0^*\omega = \int_0^1 d_{\mathbb{R}^3}\left(i_{\frac{\partial}{\partial t}} H^*\omega\right) dt$$

We shall check this now: it is simply a matter of differentiating carefully and integrating by parts

First, we check that the coefficient of $dx \wedge dy$ in $\int_0^1 d_{\mathbb{R}^3}(i_{\frac{\partial}{\partial t}} H^* \omega)$ actually is $R(x,y,z)$ when evaluated at $(x,y,z)$. For this, we note that

$\frac{\partial}{\partial x}\left(Q|_{(tx,ty,tz)}\right) = t\left(\frac{\partial Q}{\partial x}\Big|_{(tx,ty,tz)}\right)$ and similarly for other "space" derivatives of $P$, $Q$, or $R$, each evaluated at $(tx,ty,tz)$, i.e. $\frac{\partial}{\partial(x \text{ or } y \text{ or } z)}\left(P|_{(tx,ty,tz)} \text{ or } Q|_{(tx,ty,tz)} \text{ or } R|_{(tx,ty,tz)}\right)$.

Now the coefficient in $d_{\mathbb{R}^3}(i_{\frac{\partial}{\partial t}} H^*\omega)$ of $dx \wedge dy$ =

$$-\frac{\partial}{\partial y}(\text{coeff. of } dx \text{ in } i_{\frac{\partial}{\partial t}} H^*\omega) + \frac{\partial}{\partial x}(\text{coeff. of } dy \text{ in } i_{\frac{\partial}{\partial t}} H^*\omega)$$

$$= t\Big(-\frac{\partial}{\partial y}\left(z\, Q|_{(tx,ty,tz)} - y\, R|_{(tx,ty,tz)}\right) + \frac{\partial}{\partial x}\left(x\, R|_{(tx,ty,tz)} - z\, P|_{(tx,ty,tz)}\right)\Big)$$

$$= t\Big( R|_{(tx,ty,tz)} + y\frac{\partial}{\partial y}\left(R|_{(tx,ty,tz)}\right) - z\frac{\partial}{\partial y}\left(Q|_{(tx,ty,tz)}\right) + R|_{(tx,ty,tz)} + x\frac{\partial}{\partial x}\left(R|_{(tx,ty,tz)}\right) - z\frac{\partial}{\partial x}\left(P|_{(tx,ty,tz)}\right)\Big)$$

$$= 2t\, R|_{(tx,ty,tz)} + t^2 y \frac{\partial R}{\partial y}\Big|_{(tx,ty,tz)} + t^2 x \frac{\partial R}{\partial x}\Big|_{(tx,ty,tz)} + t^2 z\frac{\partial R}{\partial z}\Big|_{(tx,ty,tz)}$$

where we used $-z\frac{\partial}{\partial y}\left(Q|_{(tx,ty,tz)}\right) - z\frac{\partial}{\partial x}\left(P|_{(tx,ty,tz)}\right)$

$$= -tz\left(\frac{\partial Q}{\partial y}\Big|_{(tx,ty,tz)} + \frac{\partial P}{\partial x}\Big|_{(tx,ty,tz)}\right)$$

$$= t\, z\frac{\partial R}{\partial z}\Big|_{(tx,ty,tz)}$$

(here we use $\omega$ is closed! - we had to use it somewhere!)

Next note that $t^2 x \frac{\partial R}{\partial y}\Big|_{(tx,ty,tz)}$

$$+ t^2 y \frac{\partial R}{\partial y}\Big|_{(tx,ty,tz)} + t^2 z \frac{\partial R}{\partial z}\Big|_{(tx,ty,tz)}$$

$$= t^2 \frac{\partial}{\partial t}\left(R\big|_{(tx,ty,tz)}\right) \text{ by the Chain Rule.}$$

So $\int_0^1 (dx \wedge dy \text{ coef of } d_{\mathbb{R}^3}(i_{\frac{\partial}{\partial t}} H^* \omega)\, dt =$

$$\int_0^1 \left[2t\, R\big|_{(tx,ty,tz)} + t^2 \frac{\partial}{\partial t}\left(R\big|_{(tx,ty,tz)}\right)\right] dt$$

$$= \int_0^1 \frac{\partial}{\partial t}\left(t^2 R\big|_{(tx,ty,tz)}\right) dt = 1^2 R\big|_{(x,y,z)} - 0^2 R\big|_{(0,0,0)}$$

$$= R(x,y,z), \text{ as we hoped.}$$

The calculations that the coefficient of $dy \wedge dz$ is $P$ and of $dx \wedge dz$ is $-Q$ are similar and are left as exercises.

Note that the pattern here suggests a way to prove by direct calculation the Poincare Lemma (for $k$ forms $1 \le k \le n$) for $\mathbb{R}^n$ (or a ball in $\mathbb{R}^n$) in the case of the contraction to a point being multiplication by $t \in [0,1]$ ($k$ forms $1 \le k \le n$) We do not really need this direct proof since we have the general proof using $\mathcal{L}_X = d i_X + i_X d$, but it is interesting to note that the direct proof could be developed from this example. (One supposes that something like this was what Volterra used in the first place).