June 29: Two (Similar) Second Order Linear Equations

Mass/spring/damped oscillator

(1) $m \frac{d^2x(t)}{dt^2} + \gamma \frac{dx(t)}{dt} + k\,x(t) = F(t)$

$$\frac{d^2x}{dt^2} + \frac{\gamma}{m}\frac{dx}{dt} + \frac{k}{m}x(t) = \frac{1}{m}F(t)$$

and

(2) LRC electric circuit

$$LI'' + RI' + \frac{1}{C}I = E'(t)$$

or $$I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'$$

Homogenous solution when no damping ($\gamma = 0$ or $R = 0$):

$x(t) = \text{sine or cos}\left(\sqrt{\frac{k}{m}}\,t\right)$ (or $e^{\pm i\sqrt{\frac{k}{m}}t}$)

$I(t) = \text{sine or cos}\left(\frac{1}{\sqrt{LC}}t\right)$ (or $e^{\pm i\frac{1}{\sqrt{LC}}t}$)

Terminology "resonant frequency" $= \sqrt{\frac{k}{m}}$ (or $\frac{1}{\sqrt{LC}}$)

or sometimes $f_r = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$ or $\frac{1}{2\pi\sqrt{LC}}$ so resonant solution is sin or cos $2\pi f_r$ (to make f_r like actual frequency)

Convenient notation, with $\omega_0 = \sqrt{\frac{k}{m}}$ or $\frac{1}{\sqrt{LC}}$:

$$x''(t) + 2\zeta\omega_0 x'(t) + \omega_0^2 = \frac{1}{m}F(t)$$

where $2\omega_0\zeta = \gamma/m$ or $\zeta = \gamma/2m\sqrt{\frac{k}{m}} = \gamma/2\sqrt{km}$

Homogenous solutions (taking $F \equiv 0$): look for these in form $Ce^{\omega t}$, C complex, ω real [or could use general theorem]

Substitution gives

$$\omega^2 + 2\omega\zeta\omega_0 + \omega_0^2 = 0$$

or

$$\omega = \left(-2\omega_0\zeta \pm \sqrt{4\omega_0^2\zeta^2 - 4\omega_0^2}\right)/2$$

$$= -\omega_0\zeta \pm \sqrt{\omega_0^2(\zeta^2-1)}$$

If $\zeta = 0$ then $e^{-\omega_0\zeta t}$ is a solution.

So is $te^{-\omega_0\zeta t}$ (since root is double)

If $\zeta < 1$ so that $\sqrt{\omega_0^2(\zeta^2-1)}$ is imaginary

solution have the form

$e^{-\zeta\omega_0 t}$. sin or cos of $(\omega_0\sqrt{-\zeta^2+1})t$

("under damped"). Oscillation occurs!

If $\zeta > 1$, then solutions are real exponentials

$e^{(-\omega_0\zeta \pm \sqrt{\omega_0^2(\zeta^2-1)})t}$.

(note that both exponents are negative since

$$\omega_0\zeta > \sqrt{\omega_0\zeta^2 - \omega_0^2}$$

when $\zeta > 1$).

What happens when $F(t)$ is not zero:
Could solve by variation of parameters (complicated) but when $F(t)$ is "sinusoidal", we can solve by "undetermined coefficients": If $F(t) = F_0 e^{i\omega t}$, we look for solutions in the form $C e^{i\omega t}$ (and take imaginary part to get case $F(t) = \sin \omega t$, real part to get $F(t) = \cos \omega t$):
Want C such that (given ω, ω_0, ζ)

$$C(-\omega^2 + 2\zeta\omega_0\omega i + \omega_0^2)e^{i\omega t} = F_0 e^{i\omega t}/m$$

or
$$C(-\omega^2 + 2\zeta\omega_0\omega i + \omega_0^2) = F_0/m$$

or
$$C = \frac{F_0}{m}\left(\frac{1}{-\omega^2 + 2i\zeta\omega_0\omega + \omega_0^2}\right)$$

Then
$$\operatorname{Im}(Ce^{i\omega t}) = \frac{F_0}{m}\operatorname{Im}\left(\frac{1}{(-\omega^2 + \omega_0^2)^2 + 4\zeta^2\omega_0^2\omega^2}\cdot(-\omega^2 - 2i\zeta\omega_0\omega + \omega_0^2)(\cos\omega t + i\sin\omega t)\right)$$

$$= \frac{F_0}{m}\cdot\frac{1}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\zeta^2\omega_0^2\omega^2}}\sin(\omega t + \phi)$$

[by familiar kind of calculation (familiar I hope!) for some suitable choice of ϕ.]

$$= \frac{F_0}{m\omega}\frac{1}{Z_m}\sin(\omega t + \phi) \text{ where } Z_m = \sqrt{\frac{1}{\omega^2}(\omega_0^2 - \omega^2)^2 + (2\omega_0\zeta)^2}$$