When $\omega = 0$, this before

$$\frac{F_0}{m\sqrt{(\omega_0^2)^2}} = \frac{F_0}{m\left(\frac{k}{m}\right)} = F_0/k.$$

Why is this reasonable in physical terms? (Exercise)

When ω gets really large, the maximum amplitude gets really small. Again, why is this reasonable?

Think about these points:

When ω is small (but not 0),
ϕ is close to 0: the driving force is almost "in phase" with the amplitude.
When ω is large, the driving force is almost "180°" out of phase with the amplitude.
Why are these, too, physically reasonable?

For these, you need to compute the actual imaginary part

$$= \frac{F_0}{m}\frac{1}{\sqrt{}}\frac{(\omega_0^2 - \omega^2)\sin\omega t - 2\zeta\omega\omega_0\cos\omega t)}{\sqrt{}}$$

etc. so

$$\sin(\omega t + \phi) = -\frac{2\zeta\omega\omega_0}{\sqrt{}}\cos\omega t + \frac{(\omega_0^2 - \omega^2)}{\sqrt{}}\sin\omega t$$

so $\cos\phi = \dfrac{\omega_0^2 - \omega^2}{\sqrt{}}$, $\sin\phi = \dfrac{-2\omega\omega_0\zeta}{\sqrt{}}$ etc.

Interesting question: What driving frequency ω gives the largest (maximum) amplitude for $x(t)$?

Answer: The ω that minimizes

$m\omega Z_m$ or equivalently minimizes

$$(\omega_0^2 - \omega^2)^2 + (2\omega_0 \zeta)^2 \omega^2$$

$$= \omega_0^4 - 2\omega_0^2\omega^2 + \omega^4 + 4\omega_0^2 \zeta^2 \omega^2$$

$$= \omega^4 + (4\omega_0^2\zeta^2 - 2\omega_0^2)\omega^2 + \omega_0^4$$

Think of this as a quadratic polynomial in ω^2. It is minimal when

$$\omega_a^2 = -\tfrac{1}{2}(\text{middle coefficient})$$

$$= \omega_0^2(1 - 2\zeta^2)$$

or $\omega = \omega_0\sqrt{(1 - 2\zeta^2)}$ (provided $\zeta < 1/\sqrt{2}$)

Note that this is less than the undamped resonance ω_0. A (seriously) underdamped system has a "resonance frequency" in this sense which is always less than the resonance frequency of the corresponding undamped frequency