Math. 135 Notes for June 27, 2011

Second order linear differential equations with constant coefficients $y'' + py' + qy = f(x)$

$y = y(x)$, p, q constants. According to last time, it suffices to find y_1, y_2 independent solutions of $y'' + py' + qy = 0$ and then use "variation of parameters" method ($u_1 y_1 + u_2 y_2$ etc.) to solve $y'' + py' + qy = f$.

Basic idea for solving $y'' + py' + qy = 0$: (p, q constants)

Look for solution in the form $y(x) = e^{\omega x}$, ω constant

Then $y'' + py' + qy = e^{\omega x}(\omega^2 + p\omega + q)$

which $= 0$ if and only if $\omega^2 + p\omega + q = 0$.

Solution: $\omega = -\frac{p}{2} \pm \frac{1}{2}\sqrt{p^2 - 4q}$

[write $\omega = a \pm bi$ in case $p^2 - 4q < 0$, $b = \sqrt{4q - p^2}$]

Case 1: $p^2 - 4q > 0$: Two ω_1, ω_2 real different

two independent solutions $y_1 = e^{\omega_1 x}$, $y_2 = e^{\omega_2 x}$

Why these are independent:

If $e^{\omega_1 x} = C e^{\omega_2 x}$, then $C = e^{(\omega_1 - \omega_2)x}$ but C constant implies $\omega_1 - \omega_2 = 0$. So if $\omega_1 \neq \omega_2$, $e^{\omega_1 x}$ & $e^{\omega_2 x}$ are independent

Second way to see independence

$$W(e^{\omega_1}, e^{\omega_2}) = \det\begin{pmatrix} e^{\omega_1 x} & e^{\omega_2 x} \\ \omega_1 e^{\omega_1 x} & \omega_2 e^{\omega_2 x}\end{pmatrix} = (\omega_2 - \omega_1)\, e^{(\omega_1 + \omega_2)x}$$

If $\omega_2 \neq \omega_1$, this is never zero. But F, G dependent $\Rightarrow$ $W(F, G) \equiv 0$. So $e^{\omega_1 x}, e^{\omega_2 x}$ are independent.

$$= e^{ax}\cos bx\,(a^2-b^2-2a^2+a^2+b^2)$$
$$+\,e^{ax}\sin bx\,(-2ab+2ab)$$
$$= e^{ax}(\cos bx)(0) + e^{ax}(\sin bx)(0) = 0.$$

Similarly for $e^{ax}\sin bx$. We could check without complex numbers, though we knew it <u>would</u> work because of complex number things.

~~Wh~~ Why are $e^{ax}\cos bx$ and $e^{ax}\sin bx$ independent?

Because $\cos bx = 0$ and $\sin bx = 1$ when $x = \pi/2b$

while $\cos bx = 1$ and $\sin bx = 0$ when $x = 0$

so neither is a constant multiple of the other!

(also $\cos bx/\sin bx$ = cotangent bx and

$\sin bx/\cos bx = \tan bx$ and cot, tan are

not constant :)

Wronskian viewpoint

$$W(e^{ax}\cos bx, e^{ax}\sin bx) = \det\begin{pmatrix} e^{ax}\cos bx & e^{ax}\sin bx \\ ae^{ax}\cos bx - be^{ax}\sin bx & ae^{ax}\sin bx + be^{ax}\cos bx \end{pmatrix}$$

$$= ae^{2ax}\cos bx\sin bx + be^{ax}\cos^2 bx$$
$$- ae^{2ax}\sin bx\cos bx + be^{2ax}\sin^2 bx = be^{2ax}$$

which is never 0 ($b \neq 0$). Note

$$W' = 2aW = -pW \quad \text{as required!}$$

Now turn to Case 2: $\omega = -\frac{p}{2} \pm 0$. Double root!

$y_1 = e^{-\frac{p}{2}x}$ is indeed a solution.

But there needs to be a second, independent solution! In this case $y_2 = x e^{-(p/2)x}$ solves the equation

Check: $y = x e^{-(\frac{p}{2})x}$

$$y' = -\frac{p}{2} x e^{-(\frac{p}{2})x} + e^{-(\frac{p}{2})x}$$

$$y'' = \left(\frac{p}{2}\right)^2 x e^{-\frac{p}{2}x} + \left(-\frac{p}{2}\right) e^{-\frac{p}{2}x} - \frac{p}{2} e^{-\frac{p}{2}x}$$

So $$\left(\frac{p}{2}\right)^2 y + p y' + y'' = x e^{-\frac{p}{2}x}\left[\left(\frac{p}{2}\right)^2 - p\left(\frac{p}{2}\right) + \left(\frac{p}{2}\right)^2\right]$$

$$+ e^{-\frac{p}{2}x}\left[p - \frac{p}{2} - \frac{p}{2}\right]$$

$$= x e^{-\frac{p}{2}x}[0] + e^{-\frac{p}{2}x}[0] \equiv 0.$$

Check independence: $x e^{-\frac{p}{2}x} = 0$ when $x = 0$ while $e^{-\frac{p}{2}x} = 1$ when $x = 0$. So, since $x e^{-(p/2)x} \not\equiv 0$, $x e^{-(\frac{p}{2})x}$ cannot be $\equiv C e^{-\frac{p}{2}x}$ (C would have to be 0!) and $e^{-(p/2)x}$ cannot be $C x e^{-(\frac{p}{2})x}$ because then $e^{-(\frac{p}{2})x}$ would be 0 at $x = 0$, which it is not.

Wronskian check

$$W\left(e^{-\frac{p}{2}x}, x e^{-\frac{p}{2}x}\right) = \begin{vmatrix} e^{-\frac{p}{2}x} & x e^{-\frac{p}{2}x} \\ -\frac{p}{2} e^{-\frac{p}{2}x} & e^{-p/2\, x} - \frac{p}{2} x e^{-(\frac{p}{2})x} \end{vmatrix}$$

$\boxed{\text{never } 0}$

$$= e^{-px}$$ [note $W' = -pW$ as required]

This solves all cases. But there is an annoying point: the $xe^{-(p/2)x}$ solution in case 2 seems to have dropped out of the sky. It is worth looking at where it came from!

Generalidea: Write $D = \frac{d}{dx}$ and define $(D+\alpha)y = Dy + \alpha y = \frac{dy}{dx} + \alpha y$. Define

$$(D+\alpha)(D+\beta) = D^2 + (\alpha+\beta)D + \alpha\beta$$

and $D^2 = \frac{d^2}{dx^2}$ so

$$[(D+\alpha)(D+\beta)]y = \frac{d^2y}{dx^2} + (\alpha+\beta)\frac{dy}{dx} + \alpha\beta y$$

Note that

$$[(D+\alpha)(D+\beta)]y = (D+\alpha)[(D+\beta)y].$$

Now we can solve

$$(D+\alpha)[(D+\beta)y] = 0$$

by setting $(D+\beta)y = H$

Then $(D+\alpha)H = 0$ so $H = Ce^{-\alpha x}$

Thus $(D+\beta)y = Ce^{-\alpha x}$

So $\frac{dy}{dx} + \beta y = Ce^{-\alpha x}$

or $(e^{\beta x}y)' = Ce^{(\beta-\alpha)x}$

Now there are two cases : $\alpha = \beta$ and $\alpha \neq \beta$.

If $\alpha = \beta$, then $(e^{\beta x} y)' = C$ so $e^{\beta x} y = Cx + C_1$

where C_1 is an integration constant.

So $y = Cxe^{-\beta x} + C_1 e^{-\beta x}$

If $\alpha \neq \beta$, then $(e^{\beta x} y)' = Ce^{(\beta-\alpha)x}$ when integrated gives $e^{\beta x} y = \frac{1}{\beta - \alpha} C e^{(\beta-\alpha)x} + C_1$

<u>or</u> $y = \frac{1}{\beta-\alpha} C e^{-\alpha x} + C_1 e^{-\beta x}$

$= C_2 e^{-\alpha x} + C_1 e^{-\beta x}$

where $C_2 = \frac{1}{\beta - \alpha} C_2$.

Actually for case 2, we are interested in

$(D + \frac{p}{2})(D + \frac{p}{2}) y = 0$ and we get

$y = C_2 e^{-p/2 x} + C_1 x e^{-\frac{p}{2} x}$

as we hoped! This explains where the x comes from!

[Note: In class I was solving $(D-\alpha)(D-\beta)y = 0$:

This is just the same: α, β just have opposite signs!]