Notes for June 28: nth order linear equations

Most of what we did for second order linear equations has a direct extension to nth order, $n \geq 3$. But some slight differences arise that need to be noted. Form of equation: $y^{(n)} + a_{n-1}(x) y^{(n-1)} + \ldots + a_1 y' + a_0 y = f(x)$.
$a_1(x)$, $a_0(x)$

Still remains true that general solution = any particular solution + general solution of homogeneous equation $y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_1 y' + a_0 y = 0$.

Also, consequently, we can divide the problem as before

Part I: Find $y_1, \ldots, y_n$ independent solutions of homogeneous equation

Part II: Find some particular solution of inhomogeneous equation.

Part I is based on observing that the vector space of solutions of the homogeneous equation is of dimension n. This can be shown as follows: Let y_j be the solution with $y_j^{(k)}(0) = 0$ if $0 \leq k \leq n-1$, $k \neq j$ and $y_j^{(j)}(0) = 1$. Here $j = 0, 1, \ldots, n-1$ and $y^{(0)} = y$ by notational convention. Then $y_0, \ldots, y_{n-1}$ are independent ($\sum_{j=0}^{n-1} c_j y_j \equiv 0 \Rightarrow$ all c_j equal 0 as one sees by, for each j, evaluating the jth derivative at 0). Also if y is any solution of the homogeneous equation, then

$$y = \sum_{j=0}^{n-1} y^{(j)}(0)\, y_j$$

— because LHS & RHS have same value & first $n-1$ derivative at 0 so Uniqueness applies.

Thus since solution space (of the homogeneous equation) has dimension n, any n independent solutions generate it as linear combinations: If $y_1, \ldots, y_n$ are any n independent solutions then every solution has the form $\sum_{j=1}^{n} c_j y_j$

(these y_j's need not have anything at all to do with the $y_0, \ldots y_n$ at the end of the previous page!).

Now there is one case when we can actually write down such independent solutions explicitly, with formulas. This is when the $a_{n-1}(x), \ldots a_1(x)$ $a_0(x)$ are all constants. "constant coefficient case"

To treat this case, associate to $y^{(n)} + a_{n-1} y^{(n-1)} + \ldots + a_0 y$ the polynomial $w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0$

[we are effectively look for solutions in the form e^{wx}: if $w^n + \ldots + a_0 = 0$, then $y = e^{wx}$ solves the homogeneous differential equation: check this!]

The polynomial $w^n + a_{n-1} w^{n-1} + \ldots + a_1 w + a_0$ $= (w - w_1) \ldots (w - w_n)$ for some $w_1, \ldots, w_n$ of complex numbers — where some can be repeated (this comes from the Fund. Theorem of Algebra: $P(w) = w^n + \ldots + a_0$ has

has a root, say ω_1: this is Fund Th of Alg. Then $\omega - \omega_1$ divides $P(\omega)$: $P(\omega) = (\omega - \omega_1)Q(\omega)$ where $Q(\omega)$ has degree $n-1$: it starts with ω^{n-1}. So the $P(\omega) = (\omega-\omega_1)(\omega-\omega_2)\dots(\omega-\omega_n)$ conclusion follows inductively, the $n=1$ case being obvious). With the $\omega_1 \dots \omega_n$ in sight (though for $n \geq 5$, there is no general formula for them), we divide them into groups, and each group has associated solutions of the homo. diff. equation.

	Assoc. Solutions
Group I: Simple real roots ω	$e^{\omega x}$
Group II: Multiple real roots ω, ω multiplicity k	$e^{\omega x}, xe^{\omega x}, \dots x^{k-1}e^{\omega x}$
Group III: Simple complex roots $a+bi$ (always has associated simple complex root $a-bi$)	$e^{ax}\cos bx$ $e^{ax}\sin bx$
Group IV Multiple complex roots $a+bi$ multiplicity k, $a-bi$ multiplicity k	$e^{ax}\cos bx, xe^{ax}\cos bx \dots \dots x^{k-1}e^{ax}\cos bx$ $e^{ax}\sin bx, xe^{ax}\sin bx, \dots, x^{k-1}e^{ax}\sin bx$

Note that since number of roots counting multiplicities is n, number of solutions listed is n. So we need only know that these solutions are independent. This is covered in detail in notes on www.math.ucla.edu/~greene Math 135 Spring 2010 "Linear Independence"

The basic idea is to look at orders of growth. Example $e^{\omega_1 x}, \ldots e^{\omega_n x}$ are independent if $\omega_1 < \omega_2 < \ldots < \omega_n$ (ω's real). Reason:

Suppose $c_1 e^{\omega_1 x} + \ldots + c_n e^{\omega_n x} = 0$. Then

$$c_1 e^{(\omega_1-\omega_n)x} + c_2 e^{(\omega_2-\omega_n)x} + \ldots + c_{n-1} e^{(\omega_{n-1}-\omega_n)x} + c_n = 0.$$

Since $\omega_i - \omega_n < 0$ $i = 1, 2 \ldots, n-1$, letting $x \to +\infty$ gives $c_n = 0$. So $c_1 e^{\omega_1 x} + \ldots + c_{n-1} e^{\omega_{n-1} x} = 0$.

Apply same reasoning to get $c_{n-1} = 0$. Continue to get all c's $= 0$. So $e^{\omega_1 x}, \ldots e^{\omega_n x}$ are independent.

Wronskian for nth order

$$W(y_1, \ldots, y_n) = \det \begin{pmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

Easy to check (via same logic as $n=2$ case):

(1) $y_1, \ldots y_n$ dependent $\Rightarrow$ $W(y_1, \ldots y_n) = 0$
(This works whether or not $y_1 \ldots y_n$ solve the homo. eq!)

(2) If $y_1 \ldots y_n$ are solutions of the homogeneous equation, then $W(y_1, \ldots y_n) = 0$ at one point $\Rightarrow$ $y_1, \ldots y_n$ are dependent. Logic: If $W = 0$ at $x = x_0$, then the vectors $(y_j(x_0), y_j'(x_0), \ldots y_j^{(n-1)}(x_0))$ $j = 1 \ldots n$ are dependent i.e. $\sum c_j (\quad) = 0$ some c_j's not all 0.

Then $\sum c_j y_j$ solves equation and has value $= 0$ at x_0 and 1st, 2nd, .. $(n-1)$st deriv $= 0$. So $\sum c_j y_j \equiv 0$ by Uniqueness.

Example: $y_1 = e^{\omega_1 x}, \ldots \ y_n = e^{\omega_n x}$

$$W(e^{\omega_1 x}, \ldots, e^{\omega_n x}) = \det \begin{pmatrix} e^{\omega_1 x} & \cdots & e^{\omega_n x} \\ \omega_1 e^{\omega_1 x} & \cdots & \omega_n e^{\omega_n x} \\ \vdots & & \vdots \\ \omega_1^{n-1} e^{\omega_1 x} & \cdots & \omega_n^{n-1} e^{\omega_n x} \end{pmatrix}$$

$$= e^{(\omega_1 + \cdots + \omega_n)x} \det \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \omega_1^{n-1} & \cdots & \omega_n^{n-1} \end{pmatrix}$$

Independence of $e^{\omega_1 x}, \ldots e^{\omega_n x}$ (already shown)
which solve $(D-\omega_1)\ldots(D-\omega_n)y = 0$ implies

$\det \begin{pmatrix} 1 & \cdots & 1 \\ \omega_1^{n-1} & \cdots & \omega_n^{n-1} \end{pmatrix} \neq 0$. This nonzero property of the det. of
the "Vandermonde matrix" can also be shown by
algebra: $\det = \prod_{j>i} (\omega_j - \omega_i)$. But here we
get this $\neq 0$ by linear independence of $e^{\omega_1 x} \ldots e^{\omega_n x}$.

Important lemma: $(D-\alpha)^n (x^n e^{\alpha x}) = n!\, e^{\alpha x}$

(proof by induction).

So $(D-\alpha)^k (x^n e^{\alpha x}) = 0$ if $k > n$

E.g $(D-\alpha)^3 (x^2 e^{\alpha x}) = 0$ etc.

This follows since $(D-\alpha)^{k-n}(D-\alpha)^n = (D-\alpha)^k$ ← positive no

So $(D-\alpha)^k (x^n e^{\alpha x}) = D^{k-n}(D-\alpha)^n (x^n e^{\alpha x}) = n!\,(D-\alpha)^{k-n} e^{\alpha x} = 0$.