Solutions of Sample Problems: Part I,
ultra-detailed discussion of problems 1 & 2.

1 (1) $y' - y = 0$ $Dy - y = 0$ $P(\lambda) = \lambda - 1$.
Solution: $y = Ce^x$, C constant. These are only solutions since $C =$ given $y(0)$ value solves initial value problem (at $x=0$), hence gives general solution by existence and uniqueness theorem. Ans. $y = Ce^x$

(2) $y'' - y = 0$, $D^2 y - y = 0$ $P(\lambda) = \lambda^2 - 1 = (\lambda+1)(\lambda-1)$
solutions e^x, e^{-x} general solution $C_1 e^x + C_2 e^{-x}$.
These are only solutions because $y = C_1 e^x + C_2 e^{-x}$ has $y(0) = C_1 + C_2$, $y'(0) = C_1 - C_2$ so $y(0), y'(0)$ can be arbitrarily specified: $C_1 + C_2 = A$, $C_1 - C_2 = B$ is always solvable for C_1, C_2 given A, B because

$\det\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} = -2 \neq 0$. Ans. $y = C_1 e^x + C_2 e^{-x}$

(3) $y''' - y = 0$ $D^3 y - y = 0$ $P(\lambda) = \lambda^3 - 1$ so
$P(\lambda) = (\lambda - 1)(\lambda^2 + \lambda + 1)$, roots $\lambda = 1$, $\lambda = \frac{-1 + \sqrt{3}i}{2}$, $\lambda = \frac{-1 - \sqrt{3}i}{2}$

Associated solutions e^x, $e^{-x/2} e^{i(\sqrt{3}/2)x}$, $e^{-x/2} e^{-i(\sqrt{3}/2)x}$
or for real ones
e^x, $e^{-x/2}\cos(\frac{\sqrt{3}}{2}x)$, $e^{-x/2}\sin(\frac{\sqrt{3}}{2}x)$.
These are independent: either use general theorem on independence or note that $\begin{pmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{pmatrix}$ matrix

at 0 is nonsingular ($\det \neq 0$): second way is a little messy, good time to use independence argument:
$C_1 e^x + C_2 e^{-x/2}\cos(\frac{\sqrt{3}}{2}x) + C_3 e^{-x/2}\sin(\frac{\sqrt{3}}{2}x) \equiv 0 \Rightarrow$
$C_1 = 0$ (let $x \to +\infty$) so $e^{-x/2}(C_2 \cos\frac{\sqrt{3}}{2}x + C_3 \sin\frac{\sqrt{3}}{2}x) \equiv 0$

Know how this step works ②

$$\Rightarrow \quad C_2 \cos\left(\tfrac{\sqrt{3}}{2}x\right) + C_3 \sin\left(\tfrac{\sqrt{3}}{2}x\right) \equiv 0 \Rightarrow C_2 = 0,\ C_3 = 0.$$

(For midterm, you could just quote theorem. I summarized how the proof of the independence theorem works in this case. You could just quote the independence theorem, that if all λ's are distinct, then $e^{\lambda x}$'s are independent).

2. $\frac{d^2y}{dx^2} + k^2 y = \sin bx$. To solve this, first do

$$\frac{d^2y}{dx^2} + k^2 y = e^{ibx} \quad \text{or} \quad (D^2+k^2)y = e^{ibx}.$$

Solutions: $\qquad P(\lambda) = \lambda^2 + k^2 = (\lambda+ik)(\lambda-ik)$

(1) If $(ib)^2 + k^2 \neq 0$ (same as $P(ib) \neq 0$) roots are $\pm ik$)

then $\frac{1}{P(ib)} e^{ibx}$ is solution, $\frac{1}{P(ib)} e^{ibx} = \frac{1}{k^2-b^2} e^{ibx}$

Taking imaginary part gives $\frac{1}{k^2-b^2}\sin(bx)$ solves $\frac{d^2y}{dx^2} + k^2 y = \sin bx$. General solution:

$$= \frac{1}{k^2-b^2}\sin(bx) + C_1\cos(kx) + C_2 \sin(kx)$$

since general solution of $\frac{d^2y}{dx^2} + k^2 y = 0$ is

$C_1 \cos(kx) + C_2 \sin(kx)$.

(2) If $(ib)^2 + k^2 = 0$ then $P(\lambda) = (\lambda - ib)(\lambda + ib)$ since $P(\lambda) = \lambda^2 + k^2$ and $k = b$. Solution is then obtained (for $\frac{d^2y}{dx^2} + k^2 y = e^{ibx}$) by noting $Q(\lambda) = \lambda + ib$

(in our usual notation, $P(\lambda)=(\lambda-ib)\,Q(\lambda)$, $Q(ib)\neq 0$)

so solution is $\dfrac{1}{Q(ib)}xe^{ibx} = \dfrac{1}{2ib}xe^{ibx}$

To get solution of $\dfrac{d^2y}{dx^2}+k^2y = \sin bx$, take

imaginary part of $\dfrac{1}{2ib}xe^{ibx} = \dfrac{1}{2ib}x(\cos bx + i\sin bx)$

which has imaginary part $= -\frac{1}{2b}x\cos bx$.

Does this work? If $y = -\frac{1}{2b}x\cos bx$ then

$y' = -\frac{1}{2b}\cos bx + \frac{b}{2b}x\sin bx$ and

$y'' = +\frac{1}{2}\sin bx + \frac{1}{2}\sin bx + \frac{b}{2}x\cos bx$ So ~~[illegible]~~

since $k^2=b^2$:

$$y''+k^2y = \sin bx + \tfrac{b}{2}x\cos bx + b^2\left(-\tfrac{1}{2b}x\cos bx\right)$$
$$= \sin bx$$

as required. Note how tricky this is! The Q-method really amounts to something! It is much easier!

Alternative method for case 2: $(D^2+b^2)y = e^{ibx}$

so $(D+ib)\,[(D-ib)y] = e^{+ibx}$

so setting $G=(D-ib)y$, $(D+ib)G = e^{+ibx}$

and hence

$D(e^{ibx}G) = e^{2ibx}$ or $e^{+ibx}G = \dfrac{1}{2ib}e^{2ibx}+C$ or

$G = \dfrac{1}{2ib}e^{ibx} + Ce^{-ibx}$. Then $(D-ib)y = \dfrac{1}{2ib}e^{ibx} + Ce^{-ibx}$

or

$$D\left(e^{-ibx}\,y\right) = \frac{1}{2ib} + C\,e^{-2ibx}$$

or (integrating both sides)

$$e^{-ibx}\,y = \frac{1}{2ib}\,x + \frac{C}{-2ibx}\,e^{-2ibx} + C_2$$

or

$$y = \frac{1}{2ib}\,x\,e^{+ibx} + \underbrace{\left(\frac{C}{-2ibx}\right)e^{-ibx} + C_2\,e^{ibx}}_{\text{general solution of homogeneous equation}}$$

And again taking imaginary parts gives

$$y = -\frac{1}{2b}\,x\cos bx \quad + \text{ general real solution of homogeneous equation.}$$

Notice that this second method is much longer and trickier than the P, Q method used to begin with! It did work, but it was messy.

Physical interpretation of $b = k$ case: The equation described an undamped mass/spring oscillator which was being driven at its "natural resonance" = the way it oscillates when set in motion but then no force applied. Driving the oscillator at this frequency results in larger and larger amplitudes: $-\frac{1}{2b}\,x\cos bx$ as $x \to +\infty$ is unbounded.