Solving $\frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + p_n y = f(x)$

When p's are constants and $f(x) = e^{ax}$

[Here a can be a complex number!]

Write $\underline{P}(\lambda) = \lambda^n + p_1 \lambda^{n-1} + \dots + p_n$ (polynomial of degree n)

Cases: (1) If $\underline{P}(a) \neq 0$, then $y = \frac{1}{\underline{P}(a)} e^{ax}$

solves the equation.

Proof is by direct substitution: $P(D) e^{ax} = P(a) e^{ax}$.

(2) If $\underline{P}(a) = 0$, then let $k > 0$ be the largest k such that $(\lambda - a)^k$ divides [largest power of $(\lambda - a)$] into $\underline{P}(\lambda)$, so $\underline{P}(\lambda) = (\lambda - a)^k Q(\lambda)$

where $Q(a) \neq 0$. Then

$y = \frac{1}{k!\,Q(a)} x^k e^{ax}$ solves the equation.

Proof: $P(D) = Q(D)\,(D-a)^k$. Now

$(D-a)^k (x^k e^{ax}) = k!\, e^{ax}$ (proof by induction: later).

So $Q(D)\left[(D-a)^k x^k e^{ax}\right] = k!\, Q(a)\, e^{ax}$

since $Q(D) e^{ax} = Q(a) e^{ax}$ as in case (1).

Induction proof that $(D-a)^k (x^k e^{ax}) = k!\, e^{ax}$, $k \geq 1$

$k=1$ case is direct calculation $(D-a)(x e^{ax}) = e^{ax} + ax e^{ax} - ax e^{ax} = e^{ax}$.

Induction step: $(D-a)^{k+1} (x^{k+1} e^{ax}) = (D-a)^k \left[(D-a)(x^{k+1} e^{ax})\right]$

$= (D-a)^k \left[(k+1) x^k e^{ax} + x^{k+1} a e^{ax} - a x^{k+1} e^{ax}\right]$

$= (k+1)\left[(D-a)^k (x^k e^{ax})\right] = (k+1) k!\, e^{ax} = (k+1)!\, e^{ax}$

assuming k case works, $k \geq 1$ $\square$