Notes for June 22, 2011 Math. B5

Second order linear differential equations:

$$y''(x) + p(x)y'(x) + q(x)y(x) = f(x)$$, f given

p, q given

y to be found.

Crucial observation: If y_1, y_2 both solve this, then $y = y_1 - y_2$ solves

$$y''(x) + p(x)y'(x) + q(x)y(x) = 0.$$

Proof: Compute $(y_1-y_2)'' + p(x)(y_1-y_2)' + q(x)(y_1-y_2)$

$$= (y_1'' + p(x)y_1' + q(x)y) - (y_2'' + p(x)y_2' + q(x)y_2) = f(x) - f(x) = 0$$

$\square$.

So: general solution of $y'' + py' + qy = f$ can be found in two steps:

1. Find the general solution of $y'' + py' + qy = 0$
2. Find <u>one</u> solution of $y'' + py' + qy = f$, call it y_{part}

Then general sol of $y'' + py' + qy = f$ is obtained by adding y_{part} to general "homogeneous" solution of step 1.

We discuss how to do these two parts. (Odd fact: If one does part 1, there is then a systematic method with formulas to do part 2).

For part 1, we use the Existence Theorem:

Choose solutions y_1 and y_2 of $y'' + py' + qy = 0$

that satisfy $y_1(0)=1, y_1'(0)=0$ and
$y_2(0)=0 \quad y_2'(0)=1$.

[Example: $y''+y=0, \quad y_1(x)=\cos x, \quad y_2(x)=\sin x$]
These exist by the Existence Theorem. Then for every y with $y''+py'+qy=0$,

$$y(x) \equiv y(0)\,y_1(x) + y'(0)\,y_2(x).$$

Reason: $y = y(0)y_1(x) + y'(0)y_2(x)$ definitely solves $y''+py'+qy=0$ because the homogeneous equation is linear: if y_1, y_2 solve it, so does $\alpha y_1 + \beta y_2$, if $\alpha, \beta \in \mathbb{R}$.
Now the value at 0 of $y(0)y_1(x) + y'(0)y_2(x)$ is $y(0)$ since $y_1(0)=1$ and $y_2(0)=0$. Also the derivative $(y(0)y_1 + y'(0)y_2)' = y(0)y_1' + y'(0)y_2'$ at $x=0$ is equal to $y'(0)$ because $y_1'(0)=0$, & $y_2'(0)=1$. So $y(0)y_1(x) + y'(0)y_2(x)$ has the same value and first derivative at 0 as does y. By the Uniqueness Theorem,

$$y(x) = y(0)\,y_1(x) + y'(0)\,y_2(x) \quad \text{all } x$$

[One writes $y \equiv y(0)y_1 + y'(0)y_2$].

In the language of linear algebra:

The solutions of $y''+py'+qy=0$ form a vector space. (If y_1, y_2 are solutions, so is $\alpha y_1+\beta y_2$). The particular y_1, y_2 chosen on the previous page generate this vector space. Moreover, y_1 and y_2 are linearly independent: $y_1(0)=1$ $y_1'(0)=0$ and $y_2(0)=0$, $y_2'(0)=1$ together make it impossible that $y_1 = C y_2$ or $y_2 = A y_1$ for constants A or C. [Think that through].

So y_1, y_2 are a generating, linearly independent set in the vector space of solutions.

The vector space has dimension 2!

This is all we have to say about part 1 for the moment.

Part 2: With y_1, y_2 as before ($y_1(0)=1$, $y'(0)=0$, $y_2(0)=0$, $y_2'(0)=1$)

we look for the (a) solution of $y''+py'+qy=f$ in the form $u_1 y_1 + u_2 y_2$ where u_1, u_2 are to be determined. Note that y_1 and y_2 are never both 0 for the same x-value. If $y_1(x), y(x)$ were both 0, then y_1 and y_2 would have to be dependent

[Think about this: $y_1'(x)$ would be $C y_2'(x)$ and then y_1 would have to $= Cy_2$, $\equiv$ so, by Uniqueness!]
Why is $y_2'(x) \neq 0$ has to be decided first!]

If $y = u_1 y_1 + u_2 y_2$ then $y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'$.

To avoid a mess, we impose the condition $u_1' y + u_2' y_2 \equiv 0$.

Then $y' = u_1 y_1' + u_2 y_2'$ and

$$y'' = u_1' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''.$$

Then calculate that

$$y'' + py' + qy = u_1' y_1' + u_2' y_2' + u_1 (y_1'' + p y_1' + q y_1) + u_2 (y_2'' + p y_2' + q y_2)$$

$$= u_1' y_1' + u_2' y_2'$$

Since y_1, y_2 solve the homogeneous equation.

So, continuing to impose $u_1' y_1 + u_2' y_2 = 0$, we get $y = u_1 y_1 + u_2 y_2$ solves $y'' + py' + qy = f$

<u>if</u> $u_1' y_1' + u_2' y_2' = f$. So to find u_1', u_2' we need to solve the simultaneous equations (for each x)

$$u_1'(x) y_1(x) + u_2'(x) y_2(x) = 0$$
$$u_1'(x) y_1'(x) + u_2'(x) y_2'(x) = f(x).$$

These equations are solvable for sure provided

$$\det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} = y_1 y_2' - y_1' y_2$$

is never 0. (Usual condition for solving linear equations in two unknowns). In fact the solution is

$$u_1' = \frac{-f(x)\, y_2(x)}{y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)}$$

$$u_2' = \frac{f(x)\, y_1(x)}{y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x)}$$

(check this).

So we need to know somehow that $y_1 y_2' - y_1' y_2$ is never 0. Name and notation:

The Wronskian of y_1 and y_2 is

$$W(x) = y_1(x)\, y_2'(x) - y_1'(x)\, y_2(x).$$

Direct calculation (using $y_2'' = -p y_2' - q y_2$ & $y_1'' = -p y_1' - q y_1$) gives $W' = -p W$. So $W = C e^{\int p}$ for some constant C. Since $W(0) = 1$, $C \neq 0$. So W is nowhere 0! So we are in business to solve for u_1', u_2' everywhere and then get u_1, u_2 by integrating.

End of part 2

Usually we cannot find y_1, y_2 by formulas. We just know they are there by the Existence Theorem. In the case of p, q constants, however, we shall be able to find y_1, y_2 by formulas. This will be the subject of the next lecture.