Complex Line Integrals

Suppose $\gamma(t) = (x(t), y(t))$ is a curve, $t \in [a,b]$, and $f(z) = u(x,y) + i\,v(x,y)$ is a function defined on a region that contains γ, i.e. that contains $\gamma([a,b])$. Then by definition

$$\oint_\gamma f(z)\,dz = \int_a^b \left(u\frac{dx}{dt} - v\frac{dy}{dt}\right) dt + i\int_a^b \left(v\,\frac{dx}{dt} + u\frac{dy}{dt}\right) dt$$

Where did this come from? Formally, we could write $f(z)dz = (u+iv)(dx+idy)$

$$= u\,dx - v\,dy + i\,(v\,dx + u\,dy).$$

We then used the real and imaginary parts to define $\oint f(z)\,dz$. This is a definition! and its motivation, not a "proof" of anything.

This definition has a very nice property: If f is holomorphic then

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma \frac{\partial f}{\partial z}\,dz$$

Proof: $\frac{\partial f}{\partial z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$. Now we use the Cauchy Riemann Equations:

$$\text{So } \oint \frac{\partial f}{\partial z}\,dz = \int_a^b \left(\frac{\partial u}{\partial x}\frac{\partial x}{dt} - \frac{\partial v}{\partial x}\frac{dy}{dt}\right) dt + i\int_a^b \frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial x}\frac{dy}{dt}\Big) dt$$

$$= \int_a^b \left(\frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{du}{dy}\frac{dy}{dt}\right) dt + i\int_a^b \left(\frac{\partial v}{\partial x}\frac{dx}{dt} + \frac{\partial v}{\partial y}\frac{dy}{dt}\right) dt$$

$= \quad u(\gamma(b)) - u(\gamma(a)) + i\,(v(\gamma(b)) - v(\gamma(a)))$

by the usual real variable facts that

$$\oint_\gamma du = u(\gamma(b)) - u(\gamma(a))$$

and

$$\oint_\gamma dv = v(\gamma(b)) - v(\gamma(a)).$$

Rearranging $u(\gamma(b)) - u(\gamma(a)) + i(\cancel{v}(b)) - v(\gamma(a)))$ to give

$$(u+iv)\big|_{\gamma(b)} - (u+iv)\big|_{\gamma(a)}$$

$= \quad f(\gamma(b)) - f(\gamma(a))$ gives what we wanted. $\square$

Notice that $\oint_\gamma \frac{\partial f}{\partial z}\,dz = f(\text{one end of }\gamma) - f(\text{other end of }\gamma)$

works only when f is holomorphic. In general one has to use also $\frac{\partial f}{\partial \bar z}$: one makes sense of

$$f(\gamma(b)) - f(\gamma(a)) = \oint_\gamma \frac{\partial f}{\partial z}\,dz + \oint_\gamma \frac{\partial f}{\partial \bar z}\,d\bar z$$

for the general case (same idea for $\uparrow$ as for $\oint f(z)\,dz$ with suitable alterations). But we won't need this more general idea.

Now let us look at $\oint f(z)\,dz$ when f is holomorphic (rather than $\oint \frac{\partial f}{\partial z}\,dz$):

$$f(z)\,dz = u\,dx - v\,dy + i\,(u\,dy + v\,dx).$$

Both these terms, real and imaginary, are

"closed" differentials [Recall that
$F\,dx + G\,dy$ is "closed" if $\frac{\partial F}{\partial y} \equiv \frac{\partial G}{\partial x}$]:
$\frac{\partial u}{\partial y} = \frac{\partial}{\partial x}(-v)$ and $\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$ ∴ CR equations again.
So if U, the region where f is defined and holomorphic is "good" in our previous sense (e.g. over and up, or good for some other reason), then for each closed curve γ in U

$$\oint_\gamma u\,dx - v\,dy = 0$$

and

$$\oint_\gamma u\,dy + v\,dx = 0$$

Thus in this case

$$\oint_\gamma f(z)\,dz = 0 .$$

This is the Cauchy Integral Theorem: Formally:

Theorem: If U is a good region and $f: U \to \mathbb{C}$ is a holomorphic function on U and if γ is a closed curve in U, then

$$\oint_\gamma f(z)\,dz = 0 .$$

This fact will play a fundamental role in what follows: It is the basic tool for developing the special properties of holomorphic functions.