Line Integral Examples & Cauchy Integral Theorem:

Cauchy Integral Theorem: If U is simply connected and $f: U \to \mathbb{C}$ is holomorphic and $\gamma : [a,b] \to U$ is a closed curve ($\gamma(b) = \gamma(a)$), then

$$\oint_\gamma f(z)\, dz = 0$$

Examples: $U = \mathbb{C}$, $f(z) = z^n$ $n \geq 0$
$\gamma(t) = \cos t + i \sin t$ $t \in [0, 2\pi]$.

$$\oint_\gamma z^n dz = \int_0^{2\pi} (\cos t + i \sin t)^n \cdot \frac{d}{dt}(\cos t + i \sin t)\, dt$$

$$= \int_0^{2\pi} (\cos t + i \sin t)^n (-\sin t + i \cos t)\, dt$$

$$= \int_0^{2\pi} (\cos t + i \sin t)^n \cdot i (\cos t + i \sin t)\, dt$$

$$= i \int_0^{2\pi} (\cos t + i \sin t)^{n+1} dt = i \int_0^{2\pi} (\cos(n+1)t + i \sin(n+1)t)\, dt$$

$= 0$ if $n \geq 0$.

Note that $\oint z^{-1} dt = i \int_0^{2\pi} 1\, dt = 2\pi i$ by same calculation: this does not contradict the Cauchy Integral Theorem because

$1/z$ is not holomorphic on a simply connected region containing the circle $\{z : |z| = 1\}$: The point 0 is a problem: $1/z$ is undefined there, not holomorphic at that point.

Note that $\oint_{\text{unit circle}} \frac{1}{z^n}\, dt = 0$ if $n > 1$.

Same calculation! But this is just "good luck". For $\frac{1}{z^n}$, $n > 1$, 0 is still a

problem point, so the Cauchy Integral Theorem does not show that $\oint \frac{1}{z^n}\, dz = 0$. It just happens to be 0 (when $n > 1$), but the theorem does not guarantee that! You have to calculate it: Since $\frac{1}{\cos t + i \sin t} = \cos t - i \sin t$

$$\left(\oint \frac{1}{z^n}\, dz = \int_0^{2\pi} (\cos t - i \sin t)^n \, i(\cos t + i \sin t)\, dt\right.$$

$$= i \int_0^{2\pi} (\cos nt - i \sin nt)(\cos t + i \sin(t))\, dt$$

$$= i \int_0^{2\pi} \cos((-n+1)t) - i \sin(-n+1)t)$$

$= 0$ if $n > 1$ (since $-n+1 \neq 0$ in that case)

Interesting application of these calculations:

If $f(z) = a_0 + a_1 z + a_2 z^2 + \ldots$

and if everything is all right for convergence and integrating term by term then:

$$\oint_{\gamma} \frac{f(z)}{z}\,dz = \sum_{n=0}^{+\infty} \oint \frac{a_n z^n}{z}\,dz = 2\pi i\, a_0 = 2\pi i\, f(0)$$

γ unit circle

since if $n \geq 1$, $\oint \frac{z^n}{z}\,dz = 0$.

So $$f(0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z}\,dz$$

unit circle $\circlearrowleft$ counterclockwise.

This can be extended: Suppose z_0 has $|z_0| < 1$. Then with $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$

$$\oint \frac{f(z)}{z - z_0}\,dz = \oint \frac{f(z)}{z} \frac{1}{1 - \frac{z_0}{z}}\,dz$$

geometric series expansion of $\frac{1}{1 - \frac{z_0}{z}}$

$$= \oint \frac{f(z)}{z} \cdot \left(1 + \left(\frac{z_0}{z}\right) + \left(\frac{z_0}{z}\right)^2 + \cdots\right) dz$$

$$= \oint \left(\frac{a_0}{z} + \frac{a_1 z_0}{z} + \frac{a_2 z_0^2}{z} + \cdots \right) dz$$

keeping only $\frac{1}{z}$ terms! after substituting in $f(z) = a_0 + a_1 z + a_2 z^2 + \cdots$

$$= 2\pi i\, a_0 + 2\pi i\, z_0 a_1 + 2\pi i\, a_2 z_0^2 + \cdots$$

$$= 2\pi i\, f(z_0). \text{ So } \boxed{f(z_0) = \frac{1}{2\pi i} \oint \frac{f(z)}{z - z_0}\,dz}$$

This is the Cauchy Integral Formula. We shall later prove it without assuming f is given by a power series.