Basic Ideas for Line Integrals

Real line integral: $\oint_\gamma F\,dx + G\,dy \overset{\text{definition}}{=}$

$$\int_a^b \left(F(x(t), y(t)) \frac{dx}{dt} + G(x(t), y(t)) \frac{dy}{dt} \right) dt$$

where $\gamma : [a,b] \to \mathbb{R}^2$ is a curve, $\gamma(t) = (x(t), y(t))$.

Fundamental facts: If F and G are defined on a region $U \subset \mathbb{R}^2$ and $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$ then always

"there exists" $\to$ $\exists\ H : U \to \mathbb{R}$ with $\frac{\partial H}{\partial x} \equiv F$ & $\frac{\partial H}{\partial y} \equiv G$ on U <u>provided</u> that U has no "holes" in the topological sense:

or

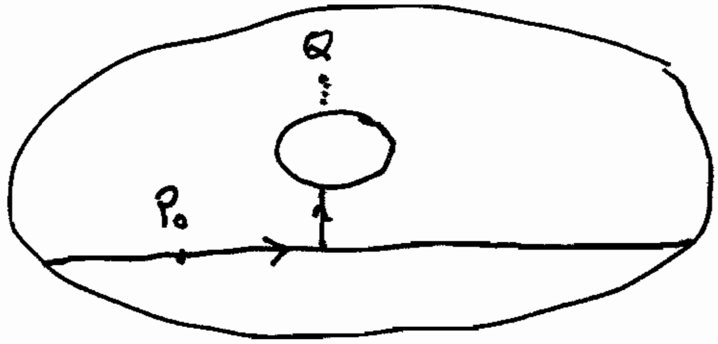
no good for some F's and G's

One condition that will guarantee that a region U is all right for this is that there is a point $P_0 \in U$ and a straight line segment, horizontal, through P_0 such that every point Q is reachable by going along L and then up or down:

good

No good for "over and up (or down)":

Cannot get to Q!



Reason region with "over and up" property works:

If $P_0 = (x_0, y_0)$ then define

$$H(x,y) = \int_{x_0}^{x} F(t, y_0)\, dt + \int_{y_0}^{y} G(x,s)\, ds$$

Then $\dfrac{\partial H}{\partial y} = G(x,y)$ (from Fundamental Theorem of Calculus)

It is less clear that $\dfrac{\partial H}{\partial x} = F(x,y)$. To check this, we use "differentiation under the integral sign" on the second term. First look at the first term

$$\frac{\partial}{\partial x}\left(\int_{x_0}^{x} F(t, y_0)\, dt = F(x, y_0).\right.$$

So we hope the second term is $F(x,y) - F(x,y_0)$ so the sum of the two is $F(x,y)$.

This actually happens, as follows

(using of course, as we must, $\dfrac{\partial G}{\partial x} = \dfrac{\partial F}{\partial y}$:

$$\frac{\partial}{\partial x}\int_{y_0}^{y} G(x,s)\,ds$$

$$= \int_{y_0}^{y} \left.\frac{\partial G}{\partial x}\right|_{(x,s)} ds = \int_{y_0}^{y} \left.\frac{\partial F}{\partial y}\right|_{(x,s)} ds$$

$$= F(x,y) - F(x,y_0).$$

So $\frac{\partial H}{\partial x} = F(x,y_0) + (F(x,y) - F(x,y_0))$

$$= F(x,y)$$

as we wanted $\square$

Corollary: If U is a "good" (over and up) region and F, G on U satisfy $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$, then $\oint_\gamma F\,dx + G\,dy = 0$ for every closed curve $\gamma \subset U$.

Proof: $H(\gamma(b)) = H(\gamma(a))$ if γ is closed $(\gamma(b) = \gamma(a))$. $\square$

Line int ④

Note that "up and over" regions (in the obvious sense) are also good! Sometimes we'll need to know that regions are good, even though they are neither "up and over" nor "over and up". For this, we can use the following often:

Fact: If U and V are good and $U \cap V$ has one piece (it is "connected") then $U \cup V$ is good.

Reason: Given $F\,dx + G\,dy$ (with $\frac{\partial F}{\partial y} = \frac{\partial G}{\partial x}$) on $U \cup V$ we can find H_1 on U with $dH_1 = F\,dx + G\,dy$ and H_2 on V with $dH_2 = F\,dx + G\,dy$. If $p \in U \cap V$, then we can change H_2 by a constant (adding a constant) to arrange that $H_1(p) = H_2(p)$. Then

$H_1 \equiv H_2$ on $U \cap V$ since H_1, H_2 have the same partial derivatives on $U \cap V$ and agree at p. Then H_1 and the adjusted H_2 fit together to give a function H on $U \cup V$ with $dH \equiv F\,dx + G\,dy$ $\square$

This will be really useful in the future. Think about it!