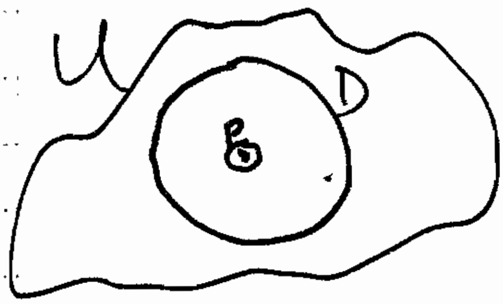
Integrals around "Isolated Singularities"

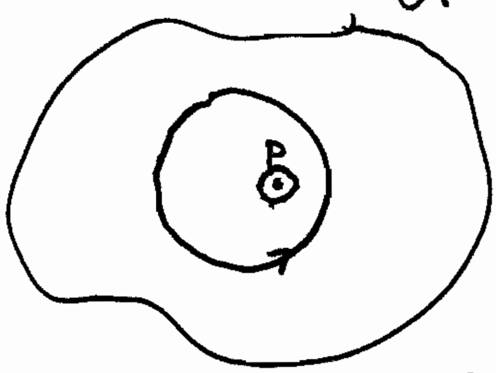
We are going to be using a lot the following idea: Suppose D is a "closed disc" $\overset{\text{def.}}{=} \{z : |z-p| \le r\}$ (some $r>0$) and U is a region that contains D. And suppose $f : U \setminus \{p\} \to \mathbb{C}$ is a holomorphic function. Then



$$\oint_{\substack{\text{boundary of } D \\ \text{circle} \\ \text{counterclockwise} \\ \circlearrowleft}} f(z)\,dz = \oint_{\substack{\text{boundary of} \\ \text{a small circle} \\ \text{around } p\,.\, \circlearrowleft}} f$$

In fact, this works even if the small circle is centered at p but the big circle is not, in other words if

U $\quad D = \{z : |z-q| \le r\}$,



$D \subset U$, f is holomorphic on $U \setminus \{p\}$ and $|p-q| < r$ so $q \in$ interior of D. Then still

$$\oint_{\substack{\text{boundary circle} \\ \text{of } D \\ \circlearrowleft}} f = \oint_{\substack{\text{boundary of} \\ \text{a small disc} \\ \text{around } p \ \circlearrowleft}} f$$

$\circlearrowleft$ means counterclockwise direction of boundary circle.

This sort of thing actually follows from what we know already together with the idea of subdividing things. To see how this works, let's draw D larger and divide it into pieces:

Fig 1

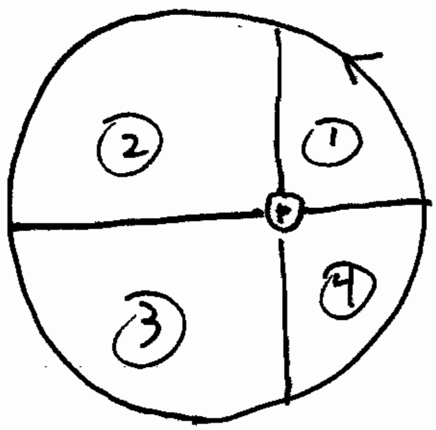
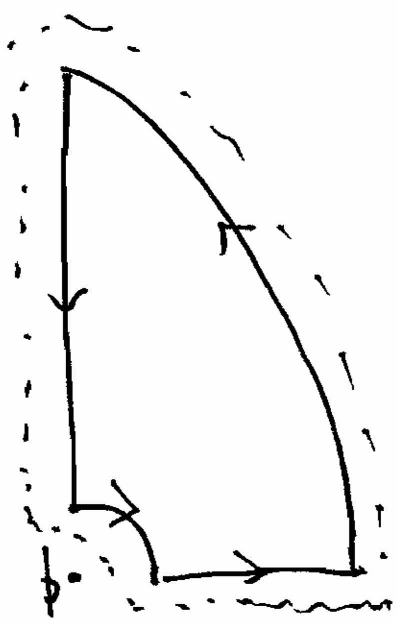


Fig 2

Fig 3

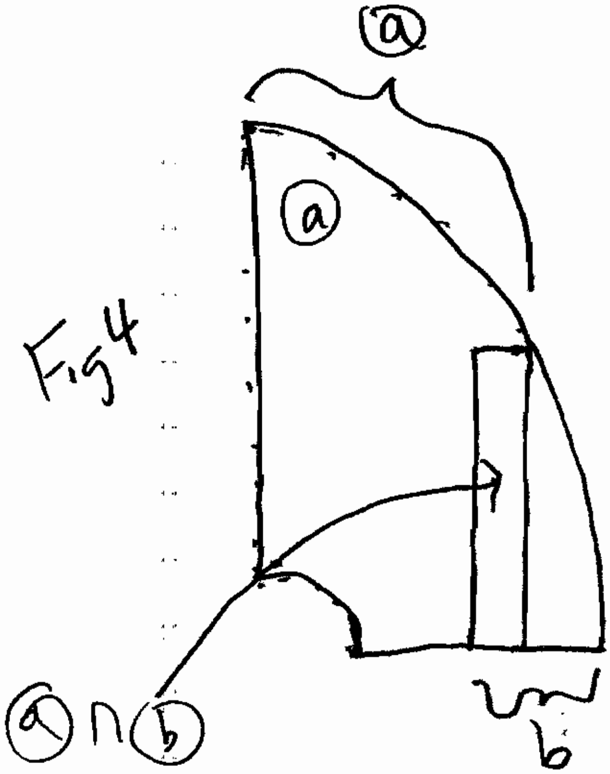


Now the $\oint f$ around the boundary curve of ① $= 0$ because ① (including its boundary) is contained in a "good" region where f is holomorphic (dotted boundary, fig 3)

You can see that this region is good because it is a union of an over and up region and an up and over region with the two of them having a connected intersection: fig. 4

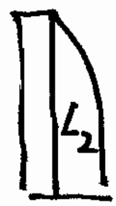
Iso sing (3)

Fig 4



(a) = 

over and up for L_1

(b) = 

up and over for L_2

This is not as complicated as it seems at first. You get used to chopping things into good pieces!

The same kind of logic shows that

$$\oint_{\text{boundary of (2)}} f = 0, \quad \text{same for (3) and (4).}$$

$$\text{But } 0 = \oint_{\circlearrowleft \text{ bdry of (1)}} f + \oint_{\circlearrowleft \text{ boundary of (2)}} f + \oint_{\circlearrowleft \text{ boundary of (3)}} f + \oint_{\circlearrowleft \text{ boundary of (4)}} f$$

$$= \oint_{\circlearrowleft \text{ boundary of D}} f - \oint_{\circlearrowleft \text{ little circle around } p} f$$

$$\text{So } \oint_{\circlearrowleft \text{ boundary of D}} f = \oint_{\text{little circle } \circlearrowleft} f$$

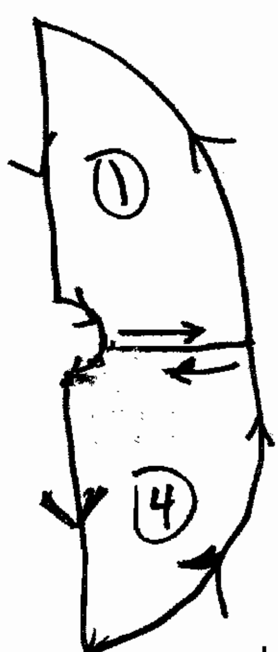
The minus sign in $\displaystyle\oint_{\text{bdry of } D} - \oint_{\substack{\text{boundary}\\ \text{of little disc}}}$

$\displaystyle = \oint_{\substack{\text{bdry of } D\\ \circlearrowleft}} - \oint_{\substack{\text{little circle } \circlearrowleft\\ \text{around } p}}$ comes

from the little circle being run

around backwards when we add up the ①, ②, ③, ④ boundary integrals.

Also, the shared boundary edges cancel out because they go way on one side, the other way for the other: look at the horizontal edge in common between ① and ④ : $\longrightarrow$ as part of boundary of ①, $\longleftarrow$ as part of boundary of ④.



There is some general theory of all this ("homology theory") but it is kind of complicated and we do not need it right now. We can just work things out by hand for our present purposes.